Neural Control 2

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Dynamic neural networks for control

The controlled nonlinear plant is given as:

$$\dot{x} = f(x,t) + u, \quad x \in \Re^n, \; u \in \Re^n$$

f is unknown.
The objective of control is to force the nonlinear system following a optimal trajectory $x^d \in \Re^r$, which is generated by

$$\dot{x}^d = \varphi(x,t)$$

The tracking error is

$$\Delta_c = x - x^d$$

Neural control

NN model

$$\dot{\hat{x}} = A\hat{x} + W_1\sigma(x) + u$$
$$\dot{x} = Ax + W_1^*\sigma(x) + u - \tilde{f}$$

Identification error

$$\Delta_i = \hat{x} - x$$

The error dynamic is

$$\dot{\Delta}_i = A\Delta_i + \tilde{W}_1\sigma(x) + \tilde{f}$$

We assume modeling error is bounded

$$\tilde{f}\Lambda_f^{-1}\tilde{f} \leq \overline{\eta}$$

Neural control

The nonlinear system can be also rewritten as

$$\dot{x} = Ax + W_1^* \sigma(x) + u - \widetilde{f}$$

or

$$\dot{x} = Ax + W_1 \sigma(x) + u - d_t$$

Becasue

$$\dot{x}^d = \varphi\left(x, t\right)$$

The tracking error dynamic is

$$\dot{\Delta}_c = Ax + W_1 \sigma(x) + u - d_t - \varphi\left(x, t\right)$$

where

$$d_t = \widetilde{f} + \tilde{W}_1 \sigma(x)$$

where the identification d_t is bounded as $\overline{d} = \sup_t \|d_t\|$.

Neural control with RNN

From the tracking error dynamic

$$\dot{\Delta}_c = Ax + W_1\sigma(x) + u - d_t - \varphi(x,t)$$

Let us select the control action u as

$$u = u_1 + u_2$$

here u_1 is direct controller, u_2 is a compensator of unmodeled dynamic d_t

$$u_1 = \varphi(x,t) - Ax^d - W_1\sigma(x)$$

and

$$\dot{\Delta}_c = A\Delta_c + u_2 + d_t$$

Sliding Mode Compensation

Define Lyapunov-like function as

$$V = \Delta_c^T P \Delta_c$$

time derivative alone $\dot{\Delta}_c = A\Delta_c + u_2 + d_t$,

$$\dot{V} = \Delta_c^T \left(A^T P + PA \right) \Delta_c + 2\Delta_c^T P u_2 + 2\Delta_c^T P d_t$$

Because

$$\begin{array}{l} \Delta_c^T \left(A^T P + PA \right) \Delta_c = -\Delta_c^T Q \Delta_c = -\left\| \Delta_c \right\|_Q^2 \\ 2\Delta_c^T P d_t \leq 2\lambda_{\max} \left(P \right) \left\| \Delta_c \right\| \left\| d_t \right\| \end{array}$$

$$\dot{V} \leq -\left\| \Delta_c \right\|_Q^2 + 2\lambda_{\max} \left(P \right) \left\| \Delta_c \right\| \left\| d_t \right\| + 2\Delta_c^T P u_2$$

Sliding Mode Compensation

We need u_2

$$2\Delta_c^T P u_2 \leq -K \left\| \Delta_c \right\|$$

Because if

$$\Delta_c sgn(\Delta_c) = \left\| \Delta_c \right\|$$

If the control u_2 has the form of

$$u_2 = -Ksgn(\Delta_c), \quad K > 0$$

then

$$2\Delta_c^T P u_2 = -2\Delta_c^T PKsgn(\Delta_c) \leq -2\lambda_{\min}(P) K \left\| \Delta_c \right\|$$

So

$$\dot{V} \leq -\left\| \Delta_c \right\|_Q^2 - 2 \left\| \Delta_c \right\| \left(\lambda_{\min}(P) K - \lambda_{\max}(P) \left\| d_t \right\| \right)$$

Sliding Mode Compensation

If we select

$$K > \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\overline{d} \tag{1}$$

then

$$\dot{V} \leq -\|\Delta_c\|_Q^2 \leq 0$$

With LaSalle lemma,

$$\lim_{t\to\infty} \Delta_c = 0$$

Exactly Compensation

Since

$$\dot{x} = Ax_t + W_1\sigma(x) + u - d_t$$
$$\dot{\hat{x}} = A\hat{x} + W_1\sigma(x) + u$$

Then

$$d_t = \left(\dot{x} - \dot{\widehat{x}}_t\right) - A\left(x - \hat{x}\right)$$

If

$$\dot{x} = f\left(x, , t\right) + u$$

is available, we can select u_2 as

$$u_2 = A\left(x - \hat{x}\right)$$
$$-\left[f\left(x, , t\right) + u - \left(A\hat{x} + W_1\sigma(x) + u\right)\right]$$

So,

$$\dot{\Delta}_c = A\Delta_c \quad \lim_{t\to\infty} \Delta_c = 0.$$

Approximate Compensation

If $\dot{x} = f(x, , t) + u$ is not available

$$\dot{x} = \frac{x_t - x_{t-\tau}}{\tau} + \delta$$

where $\delta > 0$, is the differential approximation error. Let us select the compensator as

$$u_2 = A(x - \hat{x}) - \left(\frac{x_t - x_{t-\tau}}{\tau} - \dot{\hat{x}} \right)$$

So

$$\dot{\Delta}_c = A\Delta_c + \delta$$

Approximate Compensation

Define Lyapunov-like function as

$$V = \Delta_c^T P \Delta_c$$

The time derivative is

$$\dot{V} = \Delta_c^T \left(A^T P + PA \right) \Delta_c + 2\Delta_c^T P \delta$$

$2\Delta_t^T P_2 \delta_t$ can be estimated as

$$2\Delta_c^T P \delta \leq \Delta_c^T P \Lambda P \Delta_c + \delta^T \Lambda^{-1} \delta$$

So

$$\begin{aligned} \dot{V} &\leq \Delta_c^T \left(A^T P + PA + P\Lambda P \right) \Delta_c + \delta^T \Lambda^{-1} \delta \\ &\leq -\Delta_c^T Q \Delta_c + \bar{\delta} \end{aligned}$$

Then

$$\lim \left\| \Delta_c \right\|_Q \to \bar{\delta}$$

Local Optimal Control

Because

$$V = \Delta_c^T P \Delta_c$$

time derivative alone $\dot{\Delta}_c = A\Delta_c + u_2 + d_t$,

$$\dot{V} = \Delta_c^T \left(A^T P + PA \right) \Delta_c + 2\Delta_c^T P u_2 + 2\Delta_c^T P d_t$$

$2\Delta_c^T P d_t$ can be estimated as

$$2\Delta_c^T P d_t \leq \Delta_c^T P \Lambda P \Delta_c + d_t^T \Lambda^{-1} d_t$$

Because A is stable, with the matrix Riccati equation

$$A^T P + PA + P\Lambda P + Q = 0 \qquad (2)$$

has solution.

So

$$\begin{aligned}\dot{V} &= \Delta_c^T \left(A^T P + PA\right) \Delta_c + 2\Delta_c^T P u_2 + 2\Delta_c^T P d_t \\ &\leq - \|\Delta\|_Q^2 - u_2^T R u_2 + u_2^T R u_2 + 2\Delta_c^T P u_2 + d_t^T \Lambda^{-1} d_t \\ &= - \left(\|\Delta\|_Q^2 + \|u_2\|_R^2\right) + \|u_2\|_R^2 + 2\Delta_c^T P u_2 + d_t^T \Lambda^{-1} d_t\end{aligned}$$

We define

$$\Psi\left(u_2\right) = \|u_2\|_R^2 + 2\Delta_c^T P u_2$$

then

$$\|\Delta\|_Q^2 + \|u_2\|_R^2 \leq \Psi\left(u_2\right) + d_t^T \Lambda^{-1} d_t - \dot{V}$$

Local Optimal Control

Integrating each term from 0 to T, dividing each term by T, and taking the limit, for $T \longrightarrow \infty$

$$
\begin{aligned}
&\lim_{T\rightarrow\infty} \frac{1}{T} \int_0^T \|\Delta\|_Q^2 \, dt + \lim_{T\rightarrow\infty} \frac{1}{T} \int_0^T \|u_2\|_R^2 \, dt \\
&\leq \lim_{T\rightarrow\infty} \frac{1}{T} \int_0^T \Psi\left(u_2\right) dt + \left(\lim_{T\rightarrow\infty} \frac{1}{T} \int_0^T d_t^T \Lambda^{-1} d_t dt - \lim_{T\rightarrow\infty} \int_0^T \dot{V} d
\end{aligned}
$$

$$
\leq \lim_{T\rightarrow\infty} \frac{1}{T} \int_0^T \Psi\left(u_2\right) dt + \lim_{T\rightarrow\infty} \frac{1}{T} V_0
$$

so

$$
\min\left(\|\Delta\|_Q^2 + \|u_2\|_R^2\right) \longrightarrow \min \Psi\left(u_2^d\right)
$$

Neural control: Local Optimal Control

The local optimal control is,

$$\begin{cases} \min \Psi\left(u_2\right) = \|u_2\|_R^2 + 2\Delta_c^T P u_2 \\ \text{subject:} A_0 u \leq B_0 \end{cases}$$

where A_0 and B_0 are some unknow matrices.

$$\Psi\left(u_2\right) = \|u_2\|_R^2 + 2\Delta_c^T P u_2$$

It is typical quadratic programming problem.
Without restriction of A_0 and B_0, u_2 is selected according to the linear squares optimal control law

$$u_2 = -R^{-1} P \Delta_c$$

where Riccati equation

$$A^T P + PA + P\Lambda P + Q = 0 \tag{3}$$

Discret-time

relative-degree-one system,

$$y(k) = f\left[X(k)\right] + g\left[X(k)\right]u(k)$$
$$X(k) = \left[y\left(k-1\right), \cdots y\left(k-n\right), u\left(k-1\right), \cdots u\left(k-m\right)\right]$$

where $f\left[\cdot\right]$ and $g\left[\cdot\right]$ are smooth functions, $g\left[\cdot\right]$ is bounded away from zero. In state space form

$$x_i\left(k+1\right) = x_{i+1}\left(k\right), \quad i = 1 \cdots n-1$$
$$x_n\left(k+1\right) = f\left[x\left(k\right)\right] + g\left[x\left(k\right)\right]u\left(k\right)$$

where $x\left(k\right) = \left[x_1 \cdots x_{n+m}\right]^T$,

$$x_i\left(k\right) = y\left(k-n+i-1\right), \quad i = 1 \cdots n$$
$$x_{i+n}\left(k\right) = u\left(k-m+i-1\right), \quad i = 1 \cdots m$$

$g\left[x\left(k\right)\right]$ is nonzero. Assume $g\left[x\left(k\right)\right] \geq \overline{g} > 0$, $\overline{g}$ is known positive constant.

Discret-time control

The treacking error is defined as

$$e_{n-i}(k) = x_{n-i}(k) - x^d_{n-i}(k), \quad i = 0 \cdots n-1$$

A filtered tracking error is

$$r(k) = e_n(k) + \lambda_1 e_{n-1}(k) + \cdots + \lambda_{n-1} e_1(k)$$

where $\lambda_1 \cdots \lambda_{n-1}$ are constant selected so that $\left(z^{n-1} + \lambda_1 z^{n-2} + \cdots + \lambda_{n-1}\right)$ is stable.
The dynamic of tracking error is

$$\begin{aligned} & r(k+1) = e_n(k+1) + \lambda_1 e_n(k) + \cdots + \lambda_{n-1} e_2(k) \\ & = f\left[x(k)\right] + g\left[x(k)\right] u(k) - x^d_n(k+1) \\ & + \lambda_1 e_n(k) + \cdots + \lambda_{n-1} e_2(k) \end{aligned}$$

Ideal state feedback control is

$$u(k) = \frac{1}{g[x(k)]}\left[x_n^d(k+1) - f[x(k)] + k_v r(k) - \lambda_1 e_n(k) - \cdots - \lambda_{n-1} e\right. \tag{4}$$

where $|k_v| < 1$.

The closed-loop system is

$$r(k+1) = k_v r(k)$$

In matrix form

$$E(k+1) = AE(k) + Br(k)$$

where $E(k) = [e_1(k) \cdots e_{n-1}(k)]$, $A = \begin{bmatrix} 0 & 1 & 0 \\ & & \\ -\lambda_{n-1} & \cdots & -\lambda_1 \end{bmatrix}$,

$B = [0 \cdots 01]^T$. Becasue A is stable, $r(k)$ is asymptotical stable.

If $f\left[x\left(k\right)\right]$ and $g\left[x\left(k\right)\right]$ are unknown,

$$\begin{aligned}\widehat{f}\left[x\left(k\right)\right] &= W_1\phi_1\left[x\left(k\right)\right]\\ \widehat{g}\left[x\left(k\right)\right] &= W_2\phi_2\left[x\left(k\right)\right]\end{aligned}$$

and

$$\begin{aligned}f\left[x\left(k\right)\right] &= W_1^*\phi_1\left[x\left(k\right)\right]+\widetilde{f}_1 = W_1\phi_1\left[x\left(k\right)\right]+\widetilde{f}\\ g\left[x\left(k\right)\right]u\left(k\right) &= W_2^*\phi_2\left[x\left(k\right)\right]u\left(k\right)+\widetilde{g}_1 = W_2\phi_2\left[x\left(k\right)\right]+\widetilde{g}\end{aligned}$$

The dynamic of tracking error is

$$\begin{aligned}r(k+1) = W_1^*\phi_1\left[x\left(k\right)\right]+W_2^*\phi_2\left[x\left(k\right)\right]u\left(k\right)\\ -x_n^d\left(k+1\right)+\lambda_1 e_n(k)+\cdots+\lambda_{n-1}e_2\left(k\right)+d\left(k\right)\end{aligned}$$

where $d\left(k\right)=\widetilde{f}+\widetilde{g}$.

Discret-time control

Then NN based control is

$$u(k) = [W_{2,k}\phi_2]^+ \{x_n^d(k+1) - W_{1,k}\phi_1[x(k)] \\ +k_v r(x) - \lambda_1 e_n(k) - \cdots - \lambda_{n-1} e_2(k)\}$$

where $[\cdot]^+$ stands for the pseudoinverse in Moor-Penrose sense,

$$x^+ = \frac{x^T}{\|x\|^2},\ 0^+ = 0$$

The closed-loop system is

$$\begin{aligned} & r(k+1) = W_1\phi_1[x(k)] + W_2\phi_2[x(k)]\,u(k) \\ & -x_n^d(k+1) + \lambda_1 e_n(k) + \cdots + \lambda_{n-1} e_2(k) + d(k) \\ & = k_v r(k) + d_1(k) \end{aligned}$$

where $\widetilde{W}_{1,k} = W_{1,k} - W_1^*$, $\widetilde{W}_{2,k} = W_{2,k} - W_2^*$, $d_1(k)$ is bounded

Theorem

The following gradient updating law can make tracking error $r(k)$ bounded (stable in an L_∞ sense)

$$\begin{array}{l} W_{1,k+1} = W_{1,k} - \eta_k r(k)\, \phi_1^T[x(k)] \\ W_{2,k+1} = W_{2,k} - \eta_k r(k)\, \phi_2^T[x(k)]\, u(k) \end{array}$$

where η_k satisfies

$$\eta_k = \begin{cases} \dfrac{\eta}{1 + \|\phi_1\|^2 + \|\phi_2 u\|^2} & \text{if } \beta\,\|r(k+1)\| \geq \|r(k)\| \\ 0 & \text{if } \beta\,\|r(k+1)\| \geq \|r(k)\| \end{cases}$$

here $1 \geq \eta > 0$, $\frac{1}{1+k_V} \geq \beta \geq 1$.

Proof

We select Lyapunov function as

$$L_k = \left\| \widetilde{W}_{1,k} \right\|^2 + \left\| \widetilde{W}_{2,k} \right\|^2 \tag{5}$$

where $\left\| \widetilde{W}_{1,k} \right\|^2 = \sum_{i=1}^{n} \widetilde{w}_{1,k,i}^2 = tr \left\{ \widetilde{W}_{1,k}^T \widetilde{W}_{1,k} \right\}$. From the updating law

$$\begin{array}{l}
\widetilde{W}_{1,k+1} = \widetilde{W}_{1,k} - \eta_k r(k) \phi^T [x(k)] \\
\Delta L_k = L_{k+1} - L_k \\
= \left\| \widetilde{W}_{1,k} - \eta_k r(k) \phi_1^T [x(k)] \right\|^2 - \left\| \widetilde{W}_{1,k} \right\|^2 \\
+ \left\| \widetilde{W}_{2,k} - \eta_k r(k) \phi_2^T [x(k)] u(k) \right\|^2 - \left\| \widetilde{W}_{2,k} \right\|^2 \\
= \eta_k^2 r^2(k) \left\| \phi_1 \right\|^2 + \eta_k^2 r^2(k) \left\| \phi_2 u \right\|^2 \\
-2\eta_k \left\| r(k) \phi_1^T \widetilde{W}_{1,k} \right\| - 2\eta_k \left\| r(k) \phi_2^T \widetilde{W}_{2,k} u(k) \right\|
\end{array}$$

Proof

Using

$$
\begin{aligned}
&-2\eta_k \left\| r(k)\,\phi_1^T \widetilde{W}_{1,k} \right\| - 2\eta_k \left\| r(k)\,\phi_2^T \widetilde{W}_{2,k} u(k) \right\| \\
&= -2\eta_k \left\| r(k) \right\| \left(\left\| \phi_1^T \widetilde{W}_{1,k} \right\| + \left\| \phi_2^T u(k)\, \widetilde{W}_{2,k} \right\| \right) \\
&\leq -2\eta_k \left\| r(k) \right\| \left\| \phi_1^T \widetilde{W}_{1,k} + \phi_2^T \widetilde{W}_{2,k} u(k) \right\| \\
&= -2\eta_k \left\| r(k) \right\| \left\| r(k+1) - k_v r(k) - \omega_1(k) \right\| \\
&= -2\eta_k \left\| r(k)\, r(k+1) - k_v r^2(k) - r(k)\,\omega_1(k) \right\|
\end{aligned}
$$

Proof

· if $\beta \left\| r\left(k+1\right) \right\| \geq \left\| r\left(k\right) \right\|$

$$-2\eta_k \left\| r\left(k\right) \phi_1^T \widetilde{W}_{1,k} \right\| - 2\eta_k \left\| r\left(k\right) \phi_2^T \widetilde{W}_{2,k} u\left(k\right) \right\|$$
$$\leq -\frac{2\eta_k}{\beta} \left\| r\left(k\right) \right\|^2 + 2\eta_k k_v \left\| r\left(k\right) \right\|^2 + \eta_k \left\| r\left(k\right) \right\|^2 + \eta_k \left\| \omega_1\left(k\right) \right\|^2$$

Using $0 < \eta \leq 1$, $0 \leq \eta_k \leq \eta \leq 1$, $\eta_k = \dfrac{\eta}{1 + \left\| \phi_1 \right\|^2 + \left\| \phi_2 u \right\|^2}$

Proof

$$
\begin{aligned}
\Delta L_k &= \eta_k^2 r^2(k)\left(\|\phi_1\|^2 + \|\phi_2 u\|^2\right) \\
&-\frac{2\eta_k}{\beta} r(k)^2 + 2\eta_k k_v r(k)^2 + \eta_k r(k)^2 + \eta_k \omega_1^2(k) \\
&= -\eta_k \left[\begin{array}{c} \left(\frac{2}{\beta} - 2k_v - 1\right) \\ -\eta \dfrac{\|\phi_1\|^2 + \|\phi_2 u\|^2}{1 + \|\phi_1\|^2 + \|\phi_2 u\|^2} \end{array} \right] r^2(k) + \eta_k \omega_1^2(k) \\
&\leq -\pi r^2(k) + \eta \omega_1^2(k)
\end{aligned} \tag{6}
$$

where $\pi = \dfrac{\eta}{1+\kappa}\left[\left(\dfrac{2\eta_k}{\beta} - 2k_v - 1\right) - \dfrac{\kappa}{1+\kappa}\right]$,

$\kappa = \max_k \left(\|\phi_1\|^2 + \|\phi_2 u\|^2\right)$.

Proof

Since $\frac{1}{1+k_V} \geq \beta \geq 1$, $\left(\frac{2}{\beta} - 2k_v - 1\right) > 1$, $\pi > 0$

$$n \min\left(\widetilde{w}_i^2\right) \leq V_k \leq n \max\left(\widetilde{w}_i^2\right)$$

where $n \times \min\left(\widetilde{w}_i^2\right)$ and $n \times \max\left(\widetilde{w}_i^2\right)$ are $\mathcal{K}_\infty$-functions, and $\pi e^2(k)$ is an $\mathcal{K}_\infty$-function, $\eta\zeta^2(k)$ is a $\mathcal{K}$-function. From (**??**) and (5) we know V_k is the function of $e(k)$ and $\zeta(k)$, so V_k admits the smooth ISS-Lyapunov function as in *Definition 2*. From *Theorem* 1, the dynamic of the identification error is input-to-state stable. The "INPUT" is corresponded to the second term of the last line in (6), *i.e.*, the modeling error $\zeta(k) = \varepsilon(k) + \mu(k)$, the "STATE" is corresponded to the first term of the last line in (6), *i.e.*, the identification error $e(k)$. Because the "INPUT" $\zeta(k)$ is bounded and the dynamic is ISS, the "STATE" $e(k)$ is bounded.

Proof

If $\beta \left\| r\left(k+1\right)\right\| < \left\| r\left(k\right)\right\|$, $\Delta L_k = 0$. L_k is constant, $W_{1,k}$, $W_{2,k}$ are the constants. Since $\left\| r\left(k+1\right)\right\| < \frac{1}{\beta}\left\| r\left(k\right)\right\|$, $\frac{1}{\beta} < 1$, $r\left(k\right)$ is bounded.

Now, we consider multilayer neural network(or multilayer perceptrons. MLP)

$$\begin{aligned} \widehat{f}\left[x\left(k\right)\right] &= W_{1,k}\phi_1\left[V_{1,k}x\left(k\right)\right] \\ \widehat{g}\left[x\left(k\right)\right] &= W_{2,k}\phi_2\left[V_{2,k}x\left(k\right)\right] \end{aligned} \tag{7}$$

where the weights in output layer are $V_{1,k}, V_{2,k} \in R^{m\times n}$, the weights in hidden layer are $W_{1,k}, W_{2,k} \in R^{1\times m}$. m is the dimension of the hidden layer, n is the dimension of the state. The feedback control is Then the control $u(k)$ can be defined as the following

$$u\left(k\right) = \left[W_{2,k}\phi_2\left(V_{2,k}x\left(k\right)\right)\right]^{+} \left[x_n^*\left(k+1\right) - W_{1,k}\phi_1\left(V_{1,k}x\left(k\right)\right) + k_v r\left(x\right) - \lambda_1 e_n(k) - \cdots - \lambda_{n-1}e_2\left(k\right)\right] \tag{8}$$

MLP

The closed-loop system becomes

$$r\left(k+1\right) = k_v r(k) + \widetilde{f} + \widetilde{g}u\left(k\right) \tag{9}$$

Similar as (**??**)

$$\begin{array}{l} \widetilde{f} = W_1^* \phi_1 \left[V_1^* x\left(k\right)\right] - W_{1,k}\phi_1\left[V_{1,k}x\left(k\right)\right] - \widetilde{f}_1 \\ \widetilde{g}u\left(k\right) = W_2^*\phi_2\left[V_2^* x\left(k\right)\right] u\left(k\right) - W_{2,k}\phi_2\left[V_{2,k}x\left(k\right)\right] u\left(k\right) - \widetilde{g}_1 \end{array}$$

In the case of two independent variables, a smooth function f has Taylor formula as

$$f\left(x_1, x_2\right) = \sum_{k=0}^{l-1} \frac{1}{k!}\left[\left(x_1 - x_1^0\right)\frac{\partial}{\partial x_1} + \left(x_2 - x_2^0\right)\frac{\partial}{\partial x_2}\right]_0^k f + R_l$$

MLP

The closed-loop system is

$$r(k+1) = k_v r(k) + \omega_2(k) + \widetilde{W}_{1,k}\phi_1 + W_{1,k}\phi_1' \widetilde{V}_{1,k} x + \widetilde{W}_{2,k}\phi_2 u + W_{2,k}\phi_2' \widetilde{V} \tag{10}$$

where $\omega_2(k) = \widetilde{f}_1 + \widetilde{g}_1 + R_1 + R_2$, $\|\omega_2(k)\|^2 \leq \overline{\omega}_2$.

Theorem

*If we use neuro adaptive control (8) to control nonlinear plant (**??**), the following updating law can make tracking error $r(k)$ bounded (stable in an L_∞ sense)*

$$\begin{array}{c} W_{1,k+1} = W_{1,k} - \eta_k r(k) \phi_1^T, \\ V_{1,k+1} = V_{1,k} - \eta_k r(k) r(k) \phi_1' W_{1,k}^T x^T(k) \\ W_{2,k+1} = W_{2,k} - \eta_k r(k) u\phi_1^T, \\ V_{2,k+1} = V_{2,k} - \eta_k r(k) u\phi_1' W_{2,k}^T x^T \end{array} \tag{11}$$

where η_k satisfies

$$\eta_k = \begin{cases} \dfrac{\eta}{\begin{array}{c}\|\phi_1\|^2 + \|\phi_2 u\|^2 \\ + \left\|\phi_1' W_{1,k}^T x^T\right\| + \left\|u\phi_2' W_{2,k}^T x^T\right\|\end{array}} & \text{if } \beta \|r(k+1)\| \geq \|r(k)\| \\ 0 & \text{if } \beta \|r(k+1)\| < \|r(k)\| \end{cases}$$

The average of the identification error satisfies

$$J = \limsup_{T\to\infty} \frac{1}{T} \sum_{k=1}^{T} r^2(k) \le \frac{\eta}{\pi}\overline{\omega}_2 \qquad (12)$$

where $\pi = \frac{\eta}{1+\kappa}\left[1 - \frac{\kappa}{1+\kappa}\right] > 0,$

$$\kappa = \max_k \left(\|\phi_1\|^2 + \|\phi_2 u\|^2 + \left\|\phi_1' W_{1,k}^T x^T\right\| + \left\|\phi_2' W_{2,k}^T x^T u\right\| \right)$$

$\overline{\omega}_2 = \max_k \left[\overline{\omega}_2^2(k)\right].$

The control goal is to force the system states $x(k)$ to track a linear reference model given by

$$x^d(k+1) = F\left[x^d(k)\right] \tag{13}$$

The model based trajectory error is

$$\Delta(k) = \hat{x}(k) - x^d(k)$$

The real tracking error is

$$\Delta^d(k) = x(k) - x^d(k)$$

The control object is

$$J_{\min} = \min_{u(k)} J, \;\; J = \left\|x(k) - x^d(k)\right\|^2$$

Recurrent neural networks for control

For any $\eta > 0$,

$$J \leq (1+\eta)\left\|x(k)-\hat{x}(k)\right\|^2 + \left(1+\eta^{-1}\right)\left\|\hat{x}(k)-x^d(k)\right\|^2 \tag{14}$$

- The minimum of the term $\left\|x(k)-\hat{x}(k)\right\|^2$ has already been solved in modeling.
- Now we can reformulate the control goal to minimize the term $\left\|\hat{x}(k)-x^d(k)\right\|^2$. We note that

$$\left\|\Delta^d(k)\right\| \geq \left\|\Delta(k)\right\|$$

Recurrent neural networks for control

For simple case

$$x\left(k+1\right)=f\left[x\left(k\right)\right]+u\left(k\right) \tag{15}$$

NN

$$\hat{x}(k+1)=A\hat{x}\left(k\right)+W_{1}\phi\left[x\left(k\right)\right]+u\left(k\right) \tag{16}$$

Reference

$$x^{d}\left(k+1\right)=F\left[x^{d}\left(k\right)\right]$$

Error dynamic is

$$\Delta\left(k\right)=\hat{x}\left(k\right)-x^{d}\left(k\right)$$
$$\Delta\left(k+1\right)=A\hat{x}\left(k\right)-Ax^{d}\left(k\right)+Ax^{d}\left(k\right)+W_{1}\phi\left[x\left(k\right)\right]+u\left(k\right)-F\,[x^{d}\,(k$$
$$=A\Delta\left(k\right)+Ax^{d}\left(k\right)+W_{1}\phi\left[x\left(k\right)\right]+u\left(k\right)-F\left[x^{d}\left(k\right)\right]$$

Recurrent neural networks for control

If

$$u = Ax^d(k) + W_1\phi[x(k)] - F\left[x^d(k)\right]$$

then

$$\Delta(k+1) = A\Delta(k)$$

$$\Delta(k) \to 0$$

$$\hat{x}(k) \to x^d(k)$$

But The real tracking error is

$$\left\|\Delta^d(k)\right\| \to \bar{\xi}$$

$\bar{\xi}$ is upper bound of NN modeling error

The unknown nonline system is

$$x(k+1) = f[x(k)] + u(k) \tag{17}$$

NN is

$$\hat{x}(k+1) = A\hat{x}(k) + W_1\phi[x(k)] + u(k) \tag{18}$$

The unknown nonlinear system can be represented as

$$x(k+1) = Ax(k) + W_1^*\phi[x(k)] + u(k) + d$$

The control goal is to force the system states $x(k)$ to track a linear reference model given by

$$x^d(k+1) = F\left[x^d(k)\right] \tag{19}$$

Recurrent neural networks for control

The real tracking error is

$$\Delta^d(k) = x(k) - x^d(k)$$

The error dynamic ic

$$\Delta(k+1) = Ax(k) + W_1\phi[x(k)] + u(k) + d - F\left[x^d(k)\right]$$

$$\Delta^d(k+1) = A\Delta^d(k) + Ax^d(k) + W_1\phi[x(k)] + u(k) - F\left[x^d(k)\right] + d$$

If

$$-u_1 = Ax^d(k) + W_1\phi[x(k)] - F\left[x^d(k)\right]$$

then

$$\Delta^d(k+1) = A\Delta^d(k) + u_2 + d$$

Sliding Mode Compensation

$$u_2 = -K sgn\left[\Delta^d(k)\right], \quad K > 0$$

Local Optimal Control

Without restriction of A_0 and B_0, $u_2$is selected according to the linear squares optimal control law

$$u_2 = -R^{-1}P_k\Delta^d(k)$$

P_k is the solution of Riccati equation

$$P_{k+1} = (A - K_k)^\top P_k(A - K_k) + Q + K_k^\top RK_k$$