Lecture 3

Characteristics of linear stationary systems (LSS) as LMIs

Some properties of linear stationary systems (LSS) may be represented as LMIs, as seen below.

3.1 LSS and their transfer function

Let as consider a *linear stationary system* given by the following equations

$$\boxed{\left.\begin{array}{c} \dot{x}(t) = Ax(t) + B_w w(t), \\ x(0) = x_0 \text{ is fixed}, \\ z(t) = C_{zx} x(t) + D_{zw} w(t), \end{array}\right\}} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system at time t, $z(t) \in \mathbb{R}^m$ is its output and $w(t) \in \mathbb{R}^k$ is an external input (or noise). The matrix A is assumed to be stable and the pair (A, B_w) is controllable, or, equivalently, the *controllability grammian* W_c defined as

$$\boxed{W_c := \int_{t=0}^{\infty} e^{At} B_w B_w^{\intercal} e^{A^{\intercal} t} dt} \tag{3.2}$$

is strictly positive definite, namely, $W_c > 0$. Applying the Laplace transformation to (3.1) we found that the *transfer function* of this LSS is equal to the following matrix

$$\boxed{H(s) = C_{zx}\left(sI_{n\times n} - A\right)^{-1} B_w + D_{zw}} \tag{3.3}$$

where $s \in \mathbb{C}$ is a complex variable.

3.2 H_2 norm

The H_2 *norm* of the LSS (3.1) is defined as

$$\boxed{\|H(s)\|_2 := \sqrt{\frac{1}{2\pi}\mathrm{Tr}\left(\int\limits_{\omega=0}^{\infty} H(j\omega)\, H^*(j\omega)\, d\omega\right)}.} \tag{3.4}$$

Here $H^*(j\omega) := H^{\intercal}(-j\omega)$ is the complex matrix adjoined to the matrix $H(j\omega)$. The norm $\|H(s)\|_2$ is finite if and only if $D_{zw} = 0$. In this case it can be calculated as follows

$$\boxed{\|H(s)\|_2^2 = Tr(C_{zx} W_c C_{zx}^{\intercal})\,.} \tag{3.5}$$

If C_{zx} is an affine function of some matrix K, i.e., $C_{zx} = C_{zx}(K)$, then the problem of finding some K fulfilling the inequality

$$\boxed{\mathrm{Tr}(C_{zx} W_c C_{zx}^{\intercal}) \le \gamma^2} \tag{3.6}$$

(here $\gamma > 0$ is a *tolerance level* of this LSS) is really LMI since by (2.25) the inequality (3.6) can be rewritten as

$$C_{zx} W_c C_{zx}^{\intercal} < \gamma^2 Q$$

and by the Schur's complement (Theorem 1.1)

$$\boxed{Tr(Q) \le 1,\ \begin{bmatrix} \gamma^2 Q & C_{zx}(K) \\ C_{zx}^{\intercal}(K) & W_c^{-1} \end{bmatrix} > 0} \tag{3.7}$$

with a slack matrix variable Q.

3.3 Passivity and the positive-real lemma

The linear stationary system (3.1) with w_t and z_t of the same size is said to be *passive* if

$$\boxed{\int_{t=0}^{T} w^{\intercal}(t)\, z(t)\, dt \geq 0} \tag{3.8}$$

for all solutions of (3.1) (generating by all admissible $w(\cdot)$) with $x(0) = 0$ and all $T \geq 0$.

Remark 3.1 *For nonzero initial conditions $x(0) \neq 0$ the definition of passivity looks as*

$$\int_{t=0}^{T} w^{\intercal}(t)\, z(t)\, dt \geq -\gamma(x(0))$$

for all admissible w_t and z_t of the same size and all $T \geq 0$ with a nonnegative function $\gamma(x(0))$ equal to 0 when $x(0) = 0$.

Passivity (when $j\omega$ is not a pole of $H(s)$) can be equivalently expressed in terms of the transfer function (3.3), namely, (3.1) is passive if and only if(t)

$$\boxed{H(s) + H^{*}(s) = 2\,\mathrm{Re}\, H(s) \geq 0 \text{ for all } \mathrm{Re}\, s > 0,} \tag{3.9}$$

that's why the passivity property some-times is called *real-positiveness*. It is said that the system (3.1) has *dissipation* $\eta \geq 0$ if

$$\boxed{\int_{t=0}^{T} w^{\intercal}(t)\, z(t)\, dt \geq \eta \int_{t=0}^{T} w^{\intercal}(t)\, w(t)\, dt} \tag{3.10}$$

for all trajectories with $x_0 = 0$ and all $T \geq 0$.

Remark 3.2 *Evidently, if (3.1) has* ***dissipation*** *$\eta = 0$, then it is* ***passive*** *(but not inverse).*

Suppose that there exists a quadratic function $V(x) := x^{\intercal} P x$, $P > 0$, such that for all x_t and w_t, satisfying (3.1), the following inequality holds

$$\frac{d}{dt} V(x_t) - 2 w_t^{\intercal} z_t + 2\eta w_t^{\intercal} w_t \leq 0. \tag{3.11}$$

Then, integrating this inequality within $[0, T]$-interval with $x_0 = 0$ yields

$$V(x(t)) - \int_{t=0}^{T} w^{\intercal}(t) z(t) \, dt + \eta \int_{t=0}^{T} w^{\intercal}(t) w(t) \, dt \leq 0$$

and, since,

$$0 \leq V(x(T)) \leq \int_{t=0}^{T} w^{\intercal}(t) z(t) \, dt - \eta \int_{t=0}^{T} w^{\intercal}(t) w(t) \, dt,$$

we obtain (3.10). So, if (3.11) holds, then one may guarantee the η-dissipation for (3.1). Simple substitution

$$\frac{d}{dt} V(x(t)) = 2x^{\intercal}(t) P \dot{x}(t) = 2x^{\intercal}(t) P [Ax(t) + B_w w(t)] =$$

$$x^{\intercal}(t) [PA + A^{\intercal} P] x(t) + x^{\intercal}(t) [PB_w] w(t) + w^{\intercal}(t) [B_w^{\intercal} P] x(t)$$

and

$$z(t) = C_{zx} x(t) + D_{zw} w(t)$$

into (3.11) implies

$$\begin{pmatrix} x(t) \\ w(t) \end{pmatrix}^{\intercal} \begin{bmatrix} PA + A^{\intercal} P & PB_w - C_{zx}^{\intercal} \\ B_w^{\intercal} P - C_{zx} & 2\eta I_{n \times n} - (D_{zw}^{\intercal} + D_{zw}) \end{bmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \leq 0.$$

The last inequality, valid for all $x(t)$ and $w(t)$ may be equivalently expressed as the LMI:

$$\boxed{\begin{bmatrix} -PA - A^{\intercal} P & -PB_w + C_{zx}^{\intercal} \\ -B_w^{\intercal} P + C_{zx} & -2\eta I_{n \times n} + (D_{zw}^{\intercal} + D_{zw}) \end{bmatrix} \geq 0.} \tag{3.12}$$

So, if there exists a matrix $P = P^{\intercal} > 0$, satisfying (3.12), then the linear system (3.1) is η- dissipative.

Lemma 3.1 (the positive-real lemma) *Under the technical condition*

$$D_{zw}^{\intercal} + D_{zw} > 2\eta I_{n \times n} \tag{3.13}$$

the ***sufficient condition of*** η***- dissipativity*** *(3.12) is equivalent by the Schur's complement (Theorem 1.1) to the existence of a positive definite solution* P *to the following matrix Riccati inequality*

$$\boxed{\left.\begin{array}{c} PA + A^{\intercal}P + \\ \\ \left[PB_w - C_{zx}\right]\left[\left(D_{zw}^{\intercal} + D_{zw}\right) - 2\eta I_{n\times n}\right]^{-1}\left[B_w^{\intercal}P - C_{zx}^{\intercal}\right] \leq 0. \end{array}\right\}} \tag{3.14}$$

Remark 3.3 *It is possible to show that the Riccati (3.14) is feasible if and only if (3.1) is passive.*

3.4 Nonexpansivity and the bounded-real lemma

The linear stationary system (3.1) is said to be *nonexspansive* if

$$\boxed{\int_{t=0}^{T} z^{\intercal}(t)\, z(t)\, dt \leq \int_{t=0}^{T} w^{\intercal}(t)\, w(t)\, dt} \tag{3.15}$$

for all solutions of (3.1) (corresponding to all admissible $w(\cdot)$) with $x(0) = 0$ and all $T \geq 0$. *Nonexpansivity* can be equivalently expressed in terms of the transfer function (3.3), namely, (3.1) is nonexspansive if and only if the following *bounded-real condition* holds

$$\boxed{H^{*}(s)\, H(s) \leq I \text{ for all } \operatorname{Re} s > 0,} \tag{3.16}$$

that's why the bounded-real property some-times is called *nonexpansivity*. This is sometimes expressed as

$$\|H\|_{\infty} \leq 1, \tag{3.17}$$

where

$$\left.\begin{array}{c} \|H\|_{\infty} := \sup\left\{\lambda_{\max}\left(H^{*}(s)\, H(s)\right) \mid \operatorname{Re} s > 0\right\} = \\ \\ \sup\left\{\lambda_{\max}\left(H^{*}(i\omega)\, H(i\omega)\right) \mid \omega \in (-\infty, \infty)\right\}. \end{array}\right\} \tag{3.18}$$

Suppose that there exists a quadratic function $V(x) := x^{\intercal}Px$, $P > 0$, such that for all x_t and w_t, satisfying (3.1), the following inequality holds

$$\frac{d}{dt}V(x(t)) - 2w(t)^{\intercal} w(t) + 2z^{\intercal}(t)\, z(t) \leq 0 \tag{3.19}$$

Then, integrating this inequality within $[0,T]$-interval with $x(0)=0$ yields

$$V(x(t)) - \int_{t=0}^{T} w(t)^{\intercal} w(t)\,dt + \int_{t=0}^{T} z^{\intercal}(t) z(t)\,dt \le 0$$

and, since,

$$0 \le V(x(T)) \le \int_{t=0}^{T} w(t)^{\intercal} w(t)\,dt - \int_{t=0}^{T} z^{\intercal}(t) z(t)\,dt$$

we obtain (3.15). So, if (3.19) holds, then one may guarantee the nonexpansivity for (3.1). Simple substitution

$$\frac{d}{dt}V(x(t)) = 2x(t)^{\intercal} P\dot{x}(t) = 2x(t)^{\intercal} P[Ax(t) + B_w w(t)] =$$

$$x(t)^{\intercal}[PA + A^{\intercal}P]x(t) + x(t)^{\intercal}[PB_w]w(t) + w^{\intercal}(t)[B_w^{\intercal}P]x(t)$$

and

$$z(t) = C_{zx}x(t) + D_{zw}w(t)$$

into (3.19) implies

$$\begin{pmatrix} x(t) \\ w(t) \end{pmatrix}^{\intercal} \begin{bmatrix} PA + A^{\intercal}P + C_{zx}^{\intercal}C_{zx} & PB_w + C_{zx}^{\intercal}D_{zw} \\ B_w^{\intercal}P + D_{zw}^{\intercal}C_{zx} & D_{zw}^{\intercal}D_{zw} - I \end{bmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \le 0,$$

which, equivalently, may be represented as the LMI

$$\boxed{\begin{bmatrix} -PA - A^{\intercal}P - C_{zx}^{\intercal}C_{zx} & -PB_w - C_{zx}^{\intercal}D_{zw} \\ -B_w^{\intercal}P - D_{zw}^{\intercal}C_{zx} & I - D_{zw}^{\intercal}D_{zw} \end{bmatrix} \ge 0.} \tag{3.20}$$

So, if there exists a matrix $P = P^{\intercal} > 0$ satisfying (3.20), then the linear system (3.1) is nonexpansive.

Lemma 3.2 (the bounded-real lemma) *Under the technical condition*

$$D_{zw}^{\intercal}D_{zw} \ne I \tag{3.21}$$

*the **sufficient condition of nonexpansivity** (3.15) is equivalent (by the Schur's complement (Theorem 1.1)) to the existence of the positive definite solution to the following matrix Riccati inequality*

$$\boxed{\begin{gathered} PA + A^{\intercal}P + C_{zx}^{\intercal}C_{zx} + \\ [PB_w - C_{zx}^{\intercal}D_{zw}][I - D_{zw}^{\intercal}D_{zw}]^{-1}[B_w^{\intercal}P - D_{zw}^{\intercal}C_{zx}] \le 0. \end{gathered}} \tag{3.22}$$

Remark 3.4 *It is possible to show that Riccati inequality (3.22) is feasible if and only if (3.1) is nonexpansive.*

3.5 H_∞ norm

The condition

$$\boxed{\|H\|_\infty \leq \gamma, \;\; 0 < \gamma} \tag{3.23}$$

can be represented as

$$\left\| \tilde{H} \right\|_\infty \leq 1$$

with the transfer function $\tilde{H}(s)$ given by

$$\begin{array}{c} \tilde{H}(s) = \tilde{C}_{zx} \left(sI_{n\times n} - A \right)^{-1} B_w + \tilde{D}_{zw} \\ \tilde{C}_{zx} := \gamma^{-1} C_{zx}, \;\; \tilde{D}_{zx} := \gamma^{-1} D_{zx} \end{array} \tag{3.24}$$

Therefore, based on the bounded-real lemma (see (3.20)), the constraint (3.23) would be met if

$$\left[\begin{array}{cc} PA + A^\intercal P + \tilde{C}_{zx}^\intercal \tilde{C}_{zx} & PB_w + \tilde{C}_{zx}^\intercal \tilde{D}_{zw} \\ B_w^\intercal P + \tilde{D}_{zw}^\intercal \tilde{C}_{zx} & \tilde{D}_{zw}^\intercal \tilde{D}_{zw} - I \end{array} \right] =$$

$$\left[\begin{array}{cc} PA + A^\intercal P + \gamma^{-2} C_{zx}^\intercal C_{zx} & PB_w + \gamma^{-2} C_{zx}^\intercal D_{zw} \\ B_w^\intercal P + \gamma^{-2} D_{zw}^\intercal C_{zx} & \gamma^{-2} D_{zw}^\intercal D_{zw} - I \end{array} \right] \leq 0,$$

which is equivalent to the feasibility of the following LMI

$$\boxed{\begin{array}{c} \left[\begin{array}{cc} \tilde{P}A + A^\intercal \tilde{P} + C_{zx}^\intercal C_{zx} & \tilde{P}B_w + C_{zx}^\intercal D_{zw} \\ B_w^\intercal \tilde{P} + D_{zw}^\intercal C_{zx} & D_{zw}^\intercal D_{zw} - \gamma^2 I \end{array} \right] \leq 0, \\ \\ 0 < \tilde{P} = \gamma^2 P. \end{array}} \tag{3.25}$$

3.6 γ-Entropy

The γ-*entropy* for the system (3.1) with the transfer function H (3.3) is defined by the following way:

$$\boxed{I_\gamma(H) := \left\{ \begin{array}{ll} \frac{-\gamma^2}{2\pi} \int\limits_{\omega=-\infty}^{\infty} \log\det\left(I - \gamma^2 H(j\omega) H^*(j\omega) \right) d\omega & \text{if } \|H\|_\infty < \gamma \\ \infty & \text{otherwise} \end{array} \right.} \tag{3.26}$$

When $\|H\|_\infty < \gamma$, the value $I_\gamma(H)$ can be calculated as

$$\boxed{I_\gamma(H) = \mathrm{Tr}\left(B_w^\intercal P B_w\right)} \tag{3.27}$$

where P is a symmetric matrix with smallest possible maximum singular value among all solutions of the following algebraic Riccati equation

$$\boxed{PA + A^\intercal P + C_{zx}^\intercal C_{zx} + \gamma^{-2} P B_w B_w^\intercal P = 0.}$$

Therefore the γ-entropy constraint $I_\gamma(H) \leq \lambda$ is equivalent to LMI in P, namely,

$$\boxed{\begin{array}{c} \begin{bmatrix} PA + A^\intercal P & PB_w & C_{zx}^\intercal \\ B_w^\intercal P & -\gamma^2 I & 0 \\ C_{zx} & 0 & -I \end{bmatrix} \leq 0, \\ \\ \tilde{D}_{zw} = 0,\ \mathrm{Tr}\left(B_w^\intercal P B_w\right) \leq \lambda. \end{array}} \tag{3.28}$$

3.7 Stability of stationary time-delay systems

Consider a *stationary time-delay system* given by

$$\boxed{\dot{x}(t) = Ax(t) + \sum_{i=1}^{L} A_i x(t - \tau_i),} \tag{3.29}$$

where $x_t \in \mathbb{R}^n$ and $\tau_i > 0$. If the *Lyapunov-Krasovski functional*

$$\boxed{\begin{array}{c} V(x(t), t) := x^\intercal(t) P x(t) + \sum\limits_{i=1}^{L} \int\limits_{s=t-\tau_i}^{t} x_s^\intercal P_i x_s ds \\ P > 0,\ P_i > 0\ \ (i = 1, ..., L) \end{array}} \tag{3.30}$$

satisfies

$$\frac{d}{dt} V(x(t), t) < 0$$

for every $x(t)$, satisfying (3.29), then this system is *asymptotically stable*, namely,

$$x_t \to 0 \text{ as } t \to \infty.$$

This can be verified by the simple calculation:

$$\boxed{\begin{array}{c} \dfrac{d}{dt}V\left(x(t)t\right) = y^{\intercal}(t)Wy(t) \\ \\ W := \begin{bmatrix} \left[PA + A^{\intercal}P + \sum\limits_{i=1}^{L} P_i\right] & PA_1 & \cdots & PA_L \\ A_1^{\intercal}P & -P_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_L^{\intercal}P & 0 & \cdots & -P_L \end{bmatrix}, \\ \\ y^{\intercal}(t) = \left(x(t), x\left(t-\tau_1\right), \cdot\cdot\cdot, x\left(t-\tau_L\right)\right), \end{array}} \tag{3.31}$$

providing that the matrices $P > 0$, $P_i > 0$ $(i = 1, ..., L)$ satisfy for all $\tau_i > 0$ the LMI

$$W < 0.$$

3.8 Hybrid time-delay linear stability

Let us consider the following *hybrid (descriptive) time-delay* linear system given by

$$\boxed{\begin{array}{c} \dfrac{d}{dt}x_1\left(t\right) = A_0 x_1\left(t\right) + A_1 x_2\left(t-\tau\right), \\ x_2\left(t\right) = A_2 x_1\left(t\right) + A_3 x_2\left(t-\tau\right), \\ x_1\left(0\right) = x_{10},\ x_2\left(\theta\right) = \psi\left(\theta\right)\ \theta \in \left[-\tau, 0\right], \end{array}} \tag{3.32}$$

where

$$A_0 \in \mathbb{R}^{n\times n}, A_1 \in \mathbb{R}^{n\times m}, A_2 \in \mathbb{R}^{m\times n}, A_3 \in \mathbb{R}^{m\times m}$$

are the given matrices of the corresponding dimensions and $\psi : \mathbb{R}^1 \to \mathbb{R}^m$ is a function from $C\left[-\tau, 0\right]$, $\tau > 0$. Notice that the first equation in (3.32) is an ordinary differential equation and the second one is a difference equation in continuous time that justifies the name "*hybrid time -delay system*".

We are interested in finding the conditions of asymptotic stability for this system. Following to [7], let us introduce the *energetic* (Lyapunov-

Krasovskii - type) *functional*

$$\boxed{\begin{array}{c} V(x_1(t),t) := x_1^{\mathsf{T}}(t)\,Px_1(t) + \displaystyle\int_{\theta=t-\tau}^{t} x_2^{\mathsf{T}}(\theta)\,Sx_2(\theta)\,d\theta, \\ \\ 0 < P = P^{\mathsf{T}} \in \mathbb{R}^{n\times n},\ 0 < S = S^{\mathsf{T}} \in \mathbb{R}^{m\times m}. \end{array}} \tag{3.33}$$

Its derivative on the trajectories of (3.32) is as follows:

$$\frac{d}{dt}V(x_1(t),x_2) = z^{\mathsf{T}}(t)\,Wz(t), \tag{3.34}$$

where

$$z^{\mathsf{T}}(t) := (x_1(t), x_2(t-\tau)) \tag{3.35}$$

and

$$\boxed{W = \begin{bmatrix} A_0^{\mathsf{T}}P + PA_0 + A_2^{\mathsf{T}}SA_2 & PA_1 + A_2^{\mathsf{T}}SA_3 \\ A_1^{\mathsf{T}}P + A_3^{\mathsf{T}}SA_2 & A_3^{\mathsf{T}}SA_3 - S \end{bmatrix}.} \tag{3.36}$$

As it is easy to see that the existence of positive definite matrices P and S such that the following LMI holds

$$W < 0$$

for all $\tau > 0$ implies the asymptotic stability of (3.32).

3.9 Examples

Example 3.1 *Find out if the system*

$$\left.\begin{array}{c} \dot{x}(t) = Ax(t) + B_w w(t), \\ x(0) = x_0 \in \mathbb{R}^3, \\ z(t) = C_{zx}x(t) + D_{zw}w(t) \end{array}\right\} \tag{3.37}$$

is η*-dissipative for*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -6.5 & -5 \end{bmatrix},\ B_w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ C_{zx} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},\ D_{zw} = 1,\ w(t) \in \mathbb{R}^1 \tag{3.38}$$

with

$$a)\ \eta = 0.5,\ \ b)\ \eta = 0.1,\ \ c)\ \eta = 0.0\ .$$

Solution. The considered linear system is η-dissipative (see (3.12)) if

$$G_\eta(P) = \begin{bmatrix} PA + A^\intercal P & PB_w - C_{zx}^\intercal \\ B_w^\intercal P - C_{zx} & 2\eta I_{n\times n} - (D_{zw}^\intercal + D_{zw}) \end{bmatrix} \leq 0.$$

The numerical solution is based on using the YALMIP-Matlab package. For some η it is

Solution 3.1

```
P=sdpvar(n,n,'symmetric');
Gp=[P*A+A'*P P*B-Czx';B'*P-Czx 2*eta*I-Dzw'-Dzw];
F=set(P>0)+set(Gp<=0);
solvesdp(F)
double(P)
double(Gp)
```

a)

$$\eta = 0.5, \ \ P = \begin{bmatrix} 4.2058 & 2.4193 & 0.4353 \\ 2.4193 & 5.0690 & 0.5839 \\ 0.4353 & 0.5839 & 0.3859 \end{bmatrix},$$

$$G_{\eta=0.5}(P) = \begin{bmatrix} -1.7410 & 0.2089 & -0.7464 \\ 0.2089 & -2.7516 & -0.2152 \\ -0.7464 & -0.2152 & -3.0770 \end{bmatrix} < 0,$$

$$\textit{eigenvalues}: -1.351\,7, -2.7879, -3.430\,0.$$

*This means that the system is **dissipative** with* $\eta = 0.5$.

b)

$$\eta = 0.1, \ \ P = \begin{bmatrix} 3.1591 & 1.9285 & 0.3319 \\ 1.9285 & 3.8829 & 0.4506 \\ 0.3319 & 0.4506 & 0.2962 \end{bmatrix},$$

$$G_{\eta=0.1}(P) = \begin{bmatrix} -1.3274 & 0.1009 & -0.4891 \\ 0.1009 & -2.0005 & -0.1887 \\ -0.4891 & -0.1887 & -2.3570 \end{bmatrix} < 0,$$

$$\textit{eigenvalues}: -\ 1.1018, -1.9972, -2.5858.$$

*This means that the system is **dissipative** with* $\eta = 0.1$.

c)

$$\eta = 0.0,\ P = \begin{bmatrix} 3.1931 & 1.9484 & 0.3353 \\ 1.9484 & 3.9877 & 0.4591 \\ 0.3353 & 0.4591 & 0.3065 \end{bmatrix},$$

$$G_{\eta=0.0}(P) = \begin{bmatrix} -1.3412 & 0.0953 & -0.5088 \\ 0.0953 & -2.0720 & -0.1944 \\ -0.5088 & -0.1944 & -2.4532 \end{bmatrix} < 0,$$

$$eigenvalues: -1.116\,6, -2.063\,0, -2.6868.$$

This means that the system is dissipative with $\eta = 0.0$, in other words, it is ***passive****.*

Example 3.2 *Consider the following time-delay system*

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + c\dot{x}(t-h) = 0 \tag{3.39}$$

with $\dot{x}(\theta) = \varphi(\theta), \theta \in [\tau, 0]$. We need to verify if this sytem is asymptotically stable for all $h > 0$ with the parameters

$$a = 1.9,\ b = 2.3,\ c = 1.5$$

Solution. Defining

$$\bar{x}(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$$

let us represent (22.6) in the standard format (3.29):

$$\dot{\bar{x}} = A_0\bar{x} + A_1\bar{x}(t-\tau)$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2.3 & -1.9 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1.5 \end{bmatrix}.$$

If there exist positive definite matrices P_0 and P_1 such that the matrix (3.31)

$$W(P_0, P_1) = \begin{bmatrix} P_0A_0 + A_0^TP_0 + P_1 & P_0A_0 \\ A_0^TP_0 & -P_1 \end{bmatrix} < 0$$

would be negative definite, then the asymptotic stability property is guaranteed. Using Matlab SeDuMi package we obtain

$$P_0 = \begin{bmatrix} 1.8901 & 0.2018 \\ 0.2018 & 0.7921 \end{bmatrix},\ P_1 = \begin{bmatrix} 0.5033 & 0.0257 \\ 0.0257 & 1.2820 \end{bmatrix},$$

providing the eigenvalues for the matrix $W(P_0, P_1)$ equal

$$(-2.5732, \ -0.5041, \ -0.3442, \ -0.1132)$$

which means that the system (22.6) is asymptotically stable uniformely for all $\tau > 0$.

Example 3.3 *Consider the hybrid system*

$$\left.\begin{array}{c} \dfrac{d}{dt}x_1(t) = a_0 x_1(t) + a_1 x_2(t-\tau) \\ \\ x_2(t) = a_2 x_1(t) + a_3 x_2(t-\tau) \end{array}\right\} \tag{3.40}$$

with $x_1(0) = x_{10}$, $x_2(\theta) = \varphi(\theta)$ $\forall\theta \in [-\tau, 0]$ *and the parameters*

$$a_0 = -12, \ a_1 = -4, \ a_2 = -3.3, \ a_3 = -0.4 \tag{3.41}$$

Let us verify if this system is asymptotically stable for all $\tau > 0$*.*

Solution. It is sufficient to verify the existence of positive definite matrices P and S (in this case, scalars) such that the LMI (3.36)

$$W(P,S) = \begin{bmatrix} 2Pa_0 + a_2^2 S & Pa_1 + a_2 a_3 S \\ a_1 P + a_2 a_3 S & a_3^2 S - S \end{bmatrix} < 0$$

is fulfilled. Again using Matlab SeDuMi package we obtain

$$P = 0.5618, \ S = 1.1098,$$

providing the negative eigenvalues $(-1.9808, -0.3487)$ for the matrix $W(P,S)$. Therefore the system (3.40) with the parameters (3.41) is asymptotically stable for all $\tau > 0$.

Exercise 3.1 *Find out if the system*

$$\left.\begin{array}{c} \dot{x}(t) = Ax(t) + B_w w(t), \\ x(0) = x_0 \in \mathbb{R}^3, \\ z(t) = C_{zx}x(t) + D_{zw}w(t) \end{array}\right\} \tag{3.42}$$

is nonexpansive for

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -5 \end{bmatrix}, \ B_w = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \tag{3.43}$$

$$C_{zx} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \ D_{zw} = 1.$$

Exercise 3.2 *For the LTI system (3.42)-(3.43) estimate* $\|H\|_\infty$*, that is, find (estimate) the minimal* $\gamma > 0$*, satisfying*

$$\|H\|_\infty \leq \gamma,\ 0 < \gamma.$$

Exercise 3.3 *Find out if the hybrid system*

$$\left.\begin{aligned} \frac{d}{dt}x_1(t) &= a_0 x_1(t) + a_1 x_2(t-\tau) \\ x_2(t) &= a_2 x_1(t) + a_3 x_2(t-\tau) \end{aligned}\right\}$$

with $x_1(0) = x_{10},\ x_2(\theta) = \varphi(\theta)\ \forall \theta \in [-\tau, 0]$ *and the parameters*

$$a_0 = -3,\ a_1 = 2,\ a_2 = -1,\ a_3 = -0.5$$

is asymptotically stable for all $\tau > 0$.