Lecture 21

Twist and Super-twist controllers

21.1 Problem formulation

Both controllers considered in this lecture have been suggested and analyzed by A. Levant in 90-es (see references in [4] and [5]). Consider again the dynamic system

$$\dot{x} = a(t, x) + b(t, x)\, u \tag{21.1}$$

where $a : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $b : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are all argument continuous vector and matrix, respectively, and $u \in \mathbb{R}^r$ is a control action. Let $\sigma = \sigma(x) \in \mathbb{R}$ be the only measurable output *(sliding variable)* which is assumed to be twice differentiable on x fulfilling the conditions

$$\left.\begin{aligned} \sigma_x^{\intercal} b &\equiv 0, \\ a^{\intercal}\sigma_{xx} b + \sigma_x^{\intercal} a_x b &\neq 0. \end{aligned}\right\} \tag{21.2}$$

Here and below we omit the time argument dependence for simplicity. Calculating total second derivative of σ, and selecting

$$u = u(\sigma, \dot{\sigma}),$$

we get

$$\ddot{\sigma} = \frac{d}{dt}\left[\sigma_x^{\intercal}(a+bu)\right] = \frac{d}{dt}\left[\sigma_x^{\intercal}a + \underbrace{(\sigma_x^{\intercal}b)}_{0}u\right] =$$

$$\frac{d}{dt}\left[\sigma_x^{\intercal}a\right] = \sigma_x^{\intercal}\dot{a} + \left[\sigma_{xx}^{\intercal}a + \sigma_x^{\intercal}a_x\right]^{\intercal}\dot{x} =$$

$$\sigma_x^{\intercal}\dot{a} + \left[\sigma_{xx}^{\intercal}a + \sigma_x^{\intercal}a_x\right]^{\intercal}(a+bu),$$

or, in the short form

$$\boxed{\ddot{\sigma} = h(t,x) + g(t,x)\,u} \tag{21.3}$$

where

$$\boxed{\left.\begin{array}{c} h(t,x) = \ddot{\sigma}\mid_{u=0} = \sigma_x^{\intercal}\dot{a} + \left[\sigma_{xx}^{\intercal}a + \sigma_x^{\intercal}a_x\right]^{\intercal}a \\ \\ g(t,x) = \left[\sigma_{xx}^{\intercal}a + \sigma_x^{\intercal}a_x\right]^{\intercal}b. \end{array}\right\}} \tag{21.4}$$

The problem, which we are interested in, is as follows. Below we will suppose that the inequalities

$$\left.\begin{array}{c} |h(t,x)| \le C, \\ \\ 0 < K_m \le \|g(t,x)\| \le K_M \end{array}\right\} \tag{21.5}$$

hold globally.

Problem 21.1 *The task is to make the output σ vanish in finite time $t_{reach} < \infty$ and to keep $\sigma = 0$ for all $t \ge t_{reach}$, namely, to fulfill*

$$\sigma = \dot{\sigma} = 0. \tag{21.6}$$

The condition $\dot{\sigma} = 0$ for all $t \ge t_{reach}$ means exactly that, starting from that time, $\ddot{\sigma} = 0$ implying

$$\boxed{\sigma = \dot{\sigma} = \ddot{\sigma} = 0.} \tag{21.7}$$

Definition 21.1 *If the property (21.7) holds we referred to this situation as the **Second Order Sliding Mode (SOSM)**.*

Consider now two most popular control laws providing SOSM for the system (21.1).

21.2 Twist controller

21.2.1 Lyapunov function analysis

Consider now the scalar case with $n = r = 1$. Let the controller is designed as

$$\boxed{\left.\begin{array}{c} u = -r_1 \mathrm{sign}(\sigma) - r_2 \mathrm{sign}(\dot{\sigma}), \\ \\ r_1 > 0, \ r_2 > 0. \end{array}\right\}} \tag{21.8}$$

Then the differential equation (21.3) for the sliding variable σ becomes

$$\left.\begin{array}{c} \ddot{\sigma} = h(t,x) + g(t,x)\, u = \\ \\ h(t,x) - g(t,x)\left[r_1 \mathrm{sign}(\sigma) + r_2 \mathrm{sign}(\dot{\sigma})\right] \end{array}\right\} \tag{21.9}$$

Represent this dynamic in the standard form, using only the first derivative values of the new variables $z_1 := \sigma$, $z_2 := \dot{\sigma}$:

$$\left.\begin{array}{c} \dot{z}_1 = z_2, \\ \\ \dot{z}_2 = h - g\left[r_1 \mathrm{sign}(z_1) + r_2 \mathrm{sign}(z_2)\right] \end{array}\right\} \tag{21.10}$$

Consider an arbitrary absolute continuos function $V(z_1, z_2)$ and its full-time derivative

$$\begin{array}{c} \dot{V} = \frac{\partial V}{\partial z_1}\dot{z}_1 + \frac{\partial V}{\partial z_2}\dot{z}_2 = \\ \\ \frac{\partial V}{\partial z_1} z_2 + \frac{\partial V}{\partial z_2}\left(h - g\left[r_1 \mathrm{sign}(z_1) + r_2 \mathrm{sign}(z_2)\right]\right). \end{array} \tag{21.11}$$

Using bounds (21.5) in (21.11) we get

$$\left.\begin{array}{c} \dot{V} \le \frac{\partial V}{\partial z_1} z_2 + C\left|\frac{\partial V}{\partial z_2}\right| - \frac{\partial V}{\partial z_2} g\left[r_1 \mathrm{sign}(z_1) + r_2 \mathrm{sign}(z_2)\right] = \\ \\ \frac{\partial V}{\partial z_1} z_2 + C\left|\frac{\partial V}{\partial z_2}\right| - \left|\frac{\partial V}{\partial z_2}\right| \mathrm{sign}\left(\frac{\partial V}{\partial z_2}\right) g\left[r_1 \mathrm{sign}(z_1) + r_2 \mathrm{sign}(z_2)\right] = \\ \\ \frac{\partial V}{\partial z_1} z_2 + C\left|\frac{\partial V}{\partial z_2}\right| - \left|\frac{\partial V}{\partial z_2}\right|\left(g r_1 \mathrm{sign}\left(z_1 \frac{\partial V}{\partial z_2}\right) + g r_2 \mathrm{sign}\left(z_2 \frac{\partial V}{\partial z_2}\right)\right) \le \\ \\ \frac{\partial V}{\partial z_1} z_2 + C\left|\frac{\partial V}{\partial z_2}\right| - \left|\frac{\partial V}{\partial z_2}\right|\left(g_1 r_1 \mathrm{sign}\left(z_1 \frac{\partial V}{\partial z_2}\right) + g_2 r_2 \mathrm{sign}\left(z_2 \frac{\partial V}{\partial z_2}\right)\right) \end{array}\right\} \tag{21.12}$$

where

$$
\left.
\begin{array}{l}
g_1 = \begin{cases} K_m & \text{if } \operatorname{sign}\left(z_1 \frac{\partial V}{\partial z_2}\right) < 0 \\ k_m & \text{if } \operatorname{sign}\left(z_1 \frac{\partial V}{\partial z_2}\right) > 0 \\ 0 & \text{if } \quad z_1 \frac{\partial V}{\partial z_2} = 0 \end{cases}, \\
g_2 = \begin{cases} K_m & \text{if } \operatorname{sign}\left(z_2 \frac{\partial V}{\partial z_2}\right) < 0 \\ k_m & \text{if } \operatorname{sign}\left(z_2 \frac{\partial V}{\partial z_2}\right) > 0 \\ 0 & \text{if } \quad z_2 \frac{\partial V}{\partial z_2} = 0 \end{cases}.
\end{array}
\right\} \tag{21.13}
$$

Finally, we get

$$
\dot{V} \leq \frac{\partial V}{\partial z_1} z_2 +
$$

$$
\frac{\partial V}{\partial z_2} \operatorname{sign}\left(\frac{\partial V}{\partial z_2}\right) \left[C - \left(g_1 r_1 \operatorname{sign}\left(z_1 \frac{\partial V}{\partial z_2}\right) + g_2 r_2 \operatorname{sign}\left(z_2 \frac{\partial V}{\partial z_2}\right)\right)\right]
$$

or, equivalently,

$$
\dot{V} \leq \frac{\partial V}{\partial z_1} z_2 + \frac{\partial V}{\partial z_2} \gamma, \tag{21.14}
$$

where

$$
\gamma = \operatorname{sign}\left(\frac{\partial V}{\partial z_2}\right) C - g_1 r_1 \operatorname{sign}(z_1) - g_2 r_2 \operatorname{sign}(z_2) \tag{21.15}
$$

If $V = V(z_1, z_2)$ satisfies the following partial differential equations

$$
\boxed{\frac{\partial V}{\partial z_1} z_2 + \frac{\partial V}{\partial z_2} \gamma = -q V^{\rho},\ \rho \in (0,1),\ q > 0,} \tag{21.16}
$$

then by (21.14) it follows

$$
\dot{V} \leq -q V^{\rho}, \tag{21.17}
$$

or

$$
\frac{dV}{V^{\rho}} \leq -q dt \Leftrightarrow \frac{1}{1-\rho} d\left(V^{1-\rho}\right) \leq -q dt,
$$

implying the finite-time convergence, i.e.

$$
0 \leq V(z_1, z_2)^{1-\rho} \leq V(z_1(0), z_2(0))^{1-\rho} - q(1-\rho) t,
$$

so that $V(z_1(t), z_2(t)) = 0$ for any

$$
\boxed{t \geq t_{reach} = \frac{V(z_1(0), z_2(0))^{1-\rho}}{q(1-\rho)}.} \tag{21.18}
$$

21.2.2 Method of Characteristics for the Lyapunov function design

To find the function $V(z_1, z_2)$ as a soltion of (21.16) let us use the following result.

Lemma 21.1 *If an absolutely continuos positive definite function* $V(z_1, z_2)$ *satisfies the following systems of ODE*

$$\frac{dz_1}{z_2} = \frac{dz_2}{\gamma} = \frac{dV}{-qV^{\rho}} \tag{21.19}$$

for $z_1^2 + z_2^2 > 0$*, then the same function is a solution of (21.16).*

Proof. For $z_1^2 + z_2^2 > 0$ from (21.19) we have

$$dz_1 = -z_2 \frac{dV}{qV^{\rho}},\ dz_2 = -\gamma \frac{dV}{qV^{\rho}},$$

and therefore

$$dV = \frac{\partial V}{\partial z_1} dz_1 + \frac{\partial V}{\partial z_2} dz_2 =$$

$$-z_2 \frac{\partial V}{\partial z_1} \frac{dV}{qV^{\rho}} - \gamma \frac{\partial V}{\partial z_2} \frac{dV}{qV^{\rho}} =$$

$$\left(-z_2 \frac{\partial V}{\partial z_1} \frac{1}{qV^{\rho}} - \gamma \frac{\partial V}{\partial z_2} \frac{1}{qV^{\rho}}\right) dV,$$

implying

$$-z_2 \frac{\partial V}{\partial z_1} \frac{1}{qV^{\rho}} - \gamma \frac{\partial V}{\partial z_2} \frac{1}{qV^{\rho}} = 1,$$

which consides with (21.16). ■

Solving the system (21.19) of ODE, rewritten as

$$\left.\begin{array}{l} \dfrac{dz_1}{z_2} = \dfrac{dz_2}{\gamma},\ \dfrac{dV}{dz_1} = -q\dfrac{V^{\rho}}{z_2}, \\ \dfrac{dV}{dz_2} = -q\dfrac{V^{\rho}}{\gamma},\ \dfrac{dz_1}{z_2} = \dfrac{dz_2}{\gamma}, \end{array}\right\}$$

we obtain the system of two 1-st integrals ("*characteristics*"), maintaining the constant values on the trajectories of the system:

$$\left.\begin{array}{r} dz_1 = \dfrac{z_2 dz_2}{\gamma}, \\ z_1 - z_1(0) = \dfrac{\gamma^{-1}}{2}\left[z_2^2 - z_2^2(0)\right], \end{array}\right\}$$

$$\left.\begin{array}{c}
\dfrac{dV}{V^{\rho}} = -q\dfrac{dz_1}{z_2} = -q\dfrac{dz_1}{\pm\sqrt{\dfrac{2}{\gamma}\left[z_1 - z_1(0)\right] + z_2^2(0)}}, \\
\Updownarrow \\
V(z_1, z_2)^{1-\rho} - V(z_1(0), z_2(0))^{1-\rho} = \\
\mp q\int\dfrac{dz_1}{\sqrt{\dfrac{2}{\gamma}\left[z_1 - z_1(0)\right] + z_2^2(0)}} := \psi(z_1), \\
\Updownarrow \\
\varphi_1(z_1, z_2, V) := V(z_1, z_2)^{1-\rho} - \psi(z_1) = \\
V(z_1(0), z_2(0))^{1-\rho} = \text{const}_1,
\end{array}\right\}$$

$$\left.\begin{array}{c}
\dfrac{dV}{V^{\rho}} = -q\dfrac{dz_2}{\gamma}, \\
\Updownarrow \\
V(z_1, z_2)^{1-\rho} - V(z_1(0), z_2(0))^{1-\rho} = -q\dfrac{1-\rho}{\gamma}\left[z_2(t) - z_2(0)\right] \\
\Updownarrow \\
\varphi_2(z_1, z_2, V) := V^{1-\rho} + q\dfrac{1-\rho}{\gamma}z_2 = \text{const}_2 = \\
V(z_1(0), z_2(0))^{1-\rho} + q\dfrac{1-\rho}{\gamma}z_2(0)
\end{array}\right\}$$

represented as

$$\left.\begin{array}{c}
\varphi_1(z_1, z_2, V) = c_1 = \text{const}_1, \\
\varphi_2(z_1, z_2, V) = c_2 = \text{const}_2.
\end{array}\right\}$$

Since any function of constants is a constant for any function Φ, we have

$$\Phi\left(\varphi_1(z_1, z_2, V), \varphi_2(z_1, z_2, V)\right) = c = \text{const}. \tag{21.20}$$

Solving this algebraic equation with respect to the variable V we obtain

$$V = V(z_1, z_2, c).$$

The function Φ and the constant c should be selected in such a way that the function $V(z_1, z_2, c)$ would be absolutely continuos and positive definite. So, there exists a lot of functions satisfying (21.20). One of possible selections is given in the theorem below.

Theorem 21.1 (Polyakov-Poznyak [12]) *The Lyapunov function V for the twist controller (21.8), which is a solution of the ODE system (21.19),* *is as follows*

$$V(z_1, z_2) = \begin{cases} \frac{k}{4}\left(\frac{z_2}{\gamma}\operatorname{sign}(z_1) + k_0\sqrt{|z_1| + \frac{z_2^2}{2\gamma}}\right)^2 & \text{if} \quad z_1 z_2 \neq 0 \\ \frac{\bar{k}}{4} z_2^2 & \text{if} \quad z_1 = 0 \\ \frac{1}{4}|z_1| & \text{if} \quad z_2 = 0 \end{cases} \tag{21.21}$$

where

$$k_0 > 0, k = \frac{1}{k_0},$$

and $\bar{k}$ *satisfies the inequalities*

$$\frac{1}{\sqrt{2(K_m(r_1 + r_2) - C}} < \bar{k} < \frac{1}{\sqrt{2(k_m r_1 - r_2 + C}}. \tag{21.22}$$

(all constants are defined in [12]).

Notice that the Lyapunov function (21.21) has a non-quadratic form expression.

21.3 Super-Twist controller

21.3.1 Lyapunov function analysis

Let us consider the controller designed as

$$\boxed{\left.\begin{array}{c} u = -\alpha\sqrt{|\sigma|}\operatorname{sign}(\sigma) - \beta \int\limits_{\tau=0}^{t} \operatorname{sign}(\sigma)\, d\tau, \\ \\ \alpha > 0, \ \beta > 0. \end{array}\right\}} \tag{21.23}$$

Remark 21.1 *In fact, the control (21.23) is a continuous control.*

Then the dynamics (21.3) of the sliding variable σ becomes

$$\ddot{\sigma} = h - g\left[\alpha\sqrt{|\sigma|}\operatorname{sign}(\sigma) + \beta \int\limits_{\tau=0}^{t} \operatorname{sign}(\sigma)\, d\tau\right], \tag{21.24}$$

where h and g are as in (21.4). Introduce new variables

$$\left.\begin{aligned} z_1 &= \dot{\sigma} + \alpha \int_{\tau=0}^{t} g\sqrt{|\sigma|}\operatorname{sign}(\sigma)\, d\tau, \\ z_2 &= \int_{\tau=0}^{t} [h - g\beta \operatorname{sign}(z_1)]\, d\tau. \end{aligned}\right\} \tag{21.25}$$

for which the following dynamics holds

$$\left.\begin{aligned} \dot{z}_1 &= z_2 - g\alpha\sqrt{|z_1|}\operatorname{sign}(z_1), \\ \dot{z}_2 &= h - g\beta \operatorname{sign}(z_1). \end{aligned}\right\} \tag{21.26}$$

It is possible to apply the Method of Characteristic to this systems of ODE and analogously obtain the corresponding Lyapunov function (see [13]). But for the simple partial case when $g \equiv 1$ and assuming

$$|h| \le L,$$

it is possible to check directly that the function (see [11])

$$V_{OACh}(z_1, z_2) = 2\beta |z_1| + \frac{1}{2}(z_2)^2 + \frac{1}{2}\left[z_2 - \alpha\sqrt{|z_1|}\operatorname{sign}(z_1)\right]^2$$

satisfies the differential inequality (21.17)

$$\dot{V}_{OACh} \le -q V_{OACh}^{\rho}$$

with

$$\rho = \frac{1}{2},\ q = \sqrt{2\beta}\min\left\{\frac{2(\alpha\beta - L - L\alpha)}{3\alpha^2 + 4\beta}, \frac{\alpha - 4L}{1+\alpha}\right\} > 0.$$

This means that we have a finite time convergence in variables z_1, z_2 implying the same effect in variables σ and $\dot{\sigma}$ with

$$t_{reach} = \frac{\sqrt{\frac{1}{\beta}}\left(4\beta + \frac{\alpha^2}{2}\right)}{\min\left\{\frac{2(\alpha\beta - L - L\alpha)}{3\alpha^2 + 4\beta}, \frac{\alpha - 4L}{1+\alpha}\right\}} |\dot{\sigma}(0)|.$$

21.4 Super-Twist observer and differentiator

21.4.1 Super-twist observer

Consider again a "mechanical" model given in the form

$$\left.\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(x_1, x_2, t, u) + \xi(x_1, x_2, t, u), \\ x_1 &\in \mathbb{R}^n, \end{aligned}\right\} \tag{21.27}$$

where $f(x_1, x_2, t, u)$ is a known part of the model and $\xi(x_1, x_2, t, u)$ is uncertain part. Here we suppose that

$$y = x_1$$

is available on-line and x_2 should be estimated. Defsign the observer as

$$\boxed{\left.\begin{aligned} \frac{d}{dt}\hat{x}_1 &= \hat{x}_2 - v_1, \\ \frac{d}{dt}\hat{x}_2 &= f(x_1, \hat{x}_2, t, u) - v_2, \end{aligned}\right\}} \tag{21.28}$$

where the **correctors** v_1 and v_2 are as follows

$$\boxed{\left.\begin{aligned} v_1 &= \alpha \|x_1 - \hat{x}_1\|^{1/2} \operatorname{SIGN}(\hat{x}_1 - x_1), \\ v_2 &= \beta \operatorname{SIGN}(\hat{x}_1 - x_1). \end{aligned}\right\}} \tag{21.29}$$

So, the estate estimation error

$$e(t) := \hat{x}(t) - x(t) \in \mathbb{R}^{2n}$$

satisfies

$$\boxed{\left.\begin{aligned} \dot{e}_1 &= e_2 - \alpha \|x_1 - \hat{x}_1\|^{1/2} \operatorname{SIGN}(e_1), \\ \dot{e}_2 &= F - \beta \operatorname{SIGN}(\hat{x}_1 - x_1), \end{aligned}\right\}} \tag{21.30}$$

where

$$F = f(x_1, \hat{x}_2, t, u) - f(x_1, x_2, t, u) - \xi(x_1, x_2, t, u).$$

Supposing

$$\|F\| \le F^{+} < \infty,$$

we obtain the same sheme as in (21.26)

$$\boxed{\left.\begin{array}{c} \dfrac{d}{dt}\hat{x}_1 = \hat{x}_2 - \alpha \left\| x_1 - \hat{x}_1 \right\|^{1/2} \mathrm{SIGN}\left(\hat{x}_1 - x_1\right), \\ \\ \dfrac{d}{dt}\hat{x}_2 = f\left(x_1, \hat{x}_2, t, u\right) - \beta \mathrm{SIGN}\left(\hat{x}_1 - x_1\right), \end{array}\right\}} \tag{21.31}$$

providing the finite -time convergence of e to zero.

21.4.2 Super-twist differentiator

The problem consists in estimating the first derivative of asignal $\phi(t)$ based on its noisy measurement

$$y(t) = \phi(t) + \eta(t).$$

Only two assumption will be made:

- the second derivative $\ddot{\phi}(t)$ of the base signal $\phi(t)$ is uniformly bounded by a known constant L, i.e.,

$$\left|\ddot{\phi}(t)\right| \le L,$$

- the measurement noise $\eta(t)$ is uniformly bounded by δ, i.e.

$$|\eta(t)| \le \delta.$$

Setting

$$x_1(t) := \phi(t), \; x_2(t) := \dot{\phi}(t),$$

the problem is transformed into the design of an observer for the system

$$\boxed{\left.\begin{array}{c} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \ddot{\phi}(t), \\ \\ y(t) = \phi(t) + \eta(t), \end{array}\right\}} \tag{21.32}$$

based on the measured output $y(t)$ only. The signal $\ddot{\phi}(t)$ is unknown and should be considered as a perturbation. Designing the state estimates $(\hat{x}_1(t), \hat{x}_2(t))$ using the supert-twist observer (21.28) we may conclude that

$\hat{x}_2(t)$ may be considered as an estimate of $\dot{\phi}(t)$. In our case we deal with the model (21.27) where the known part $f = 0$ and the uncertain part is $\xi = \ddot{\phi}$. By (21.28) with the low-pass filter application we have

$$\boxed{\begin{array}{c} \frac{d}{dt}\hat{x}_1(t) = \hat{x}_2(t) - \alpha \left\|\phi(t) - \hat{x}_1(t)\right\|^{1/2} \mathrm{SIGN}(\hat{x}_1(t) - y(t)), \\ \frac{d}{dt}\hat{x}_2(t) = -\beta \mathrm{SIGN}(\hat{x}_1(t) - y(t)), \quad \left|\dot{\phi}(t)\right| \leq \beta, \\ \text{and low-pass filter: } \mu\dot{v}(t) + v(t) = \hat{x}_2(t), \ \mu = 0.01, \end{array}} \quad (21.33)$$

so that

$$\boxed{v(t) \simeq \dot{\phi}(t).}$$

21.5 Exercises

Exercise 21.1 *Compare the Twist and Super-twist controllers for the system*

$$\left.\begin{array}{c} \dot{x} = a(t,x) + b(t,x)u \\ x \in \mathbb{R}^n, \ u \in \mathbb{R}^r, \\ a : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \ b : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times r}, \\ n = 2, \ r = 1, \end{array}\right\} \quad (21.34)$$

where u is a control action. Let

$$a(t,x) = 0.1 \sin(2t) \ln(1 + |x|), \ b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

and

$$\sigma = \sigma(x) = x_1$$

be the only measurable output (sliding variable) which is assumed to be twice differentiable on x fulfilling the conditions

$$\left.\begin{array}{c} \sigma_x^{\intercal} b \equiv 0, \\ a^{\intercal}\sigma_{xx}b + \sigma_x^{\intercal} a_x b \neq 0, \end{array}\right\} \quad \text{(cond twist-SUperTwist)}$$

taking Twist control as

$$u = -r_1 \text{sign}(z_1) + r_2 \text{sign}(z_2), \;\; r_1, r_2 > 0$$

and Super-Twist control as

$$u = -\alpha \int_{\tau=0}^{t} \sqrt{|\sigma|} \text{sign}(\sigma)\, d\tau - \beta \text{sign}(\sigma).$$

Exercise 21.2 *Calculate numerically the derivative of the function*

$$\phi(t) = \frac{a_0 + a_1 t}{b_0 + b_1 t} \arctan t, \;\; t \geq 0,$$

for the simulation take

$$a_1 = -2, \;\; b_0 = 3, \;\; b_1 = 0.1,$$

using the Super-twist differentiator (21.33) without noise $\eta(t)$ *in measurements* ($\eta(t) = 0$)*.*

Exercise 21.3 *For the system*

$$\ddot{x} + f(x, \dot{x}, t) + \xi(x, t) = u,$$

$$x \in \mathbb{R}^2, \;\; \|\xi(x, t)\| \leq \xi^+$$
$$u: \;\; \mu \dot{u} + u = e^{-0.02t}, \;\; \mu = 0.1$$

with bounded trajectories, estimate $\dot{x}$ *using ST- observer. Take*

$$f(x, \dot{x}, t) = a_0 \dot{x} + a_1 x$$

with $a_0 = 0.1$ *and* $a_1 = 4$.