Lecture 20

ASG-version of ISM control

20.0.1 Model description and problem setting

Here we will deal with the construction of a feedback, which designing is very close to the ISM approach [1], together with the, so-called, **Averaged Sub-Gradient (ASG)** Technique [14]).

Consider the dynamic model of a Lagrangian mechanical system with n-degrees of freedom in the standard form given by the following set of differential equations:

$$D\left(q\left(t\right)\right)\ddot{q}\left(t\right)+C\left(q\left(t\right),\dot{q}\left(t\right)\right)\dot{q}\left(t\right)+G\left(q\left(t\right)\right)=\tau\left(t\right)+\vartheta\left(t\right), \tag{20.1}$$

where $q(t), \dot{q}(t) \in R^n$ are the state vectors (generalized coordinates and their velocities, $t \geq 0$), $\tau(t) \in R^n$ is a vector of external torques (control) acting to the mechanical system, and $\vartheta(t) \in R^n$ is the disturbance (or uncertainty) vector.

If we wish to resolve the tracking problem for the given nominal trajectory $q^*(t)$, then we can represent the dynamics of the controlled plant in deviation coordinates

$$\boxed{\delta\left(t\right):=q\left(t\right)-q^*(t)} \tag{20.2}$$

as follows

$$\boxed{\tilde{D}\left(\delta\left(t\right)\right)\ddot{\delta}\left(t\right)=\tau\left(t\right)+\vartheta\left(t\right)-\tilde{C}\left(\delta\left(t\right),\dot{\delta}\left(t\right)\right)\dot{\delta}\left(t\right)-\tilde{G}\left(\delta\left(t\right)\right)} \tag{20.3}$$

with

$$\tilde{D}\left(\delta\right):=D\left(\delta+q^*\right),\ \tilde{C}\left(\delta,\dot{\delta}\right):=C\left(\delta+q^*,\dot{\delta}+\dot{q}^*\right),\ \tilde{G}\left(\delta\right):=G\left(\delta+q^*\right).$$

Notice that the deviation dynamics (20.3) may be represented as (omitting the time-argument)

$$\boxed{\ddot{\delta} = \tilde{D}^{-1}(\delta)\,\tau + \tilde{D}^{-1}(\delta)\,\xi,} \tag{20.4}$$

or, equivalently, as

$$\boxed{\left.\begin{array}{c} \dot{\delta}_1 = \delta_2, \;\; \delta_1 := \delta, \\ \\ \dot{\delta}_2 = \tilde{D}^{-1}(\delta_1)\,\tau + \tilde{D}^{-1}(\delta_1)\,\xi. \end{array}\right\}} \tag{20.5}$$

20.0.2 Accepted assumptions

A1. The vector of generalized coordinate $q(t)$ and its derivative $\dot{q}(t)$ are measurable on-line during the process.

A2. The matrix $D(q)$ is supposed to be known and invertible (the usual property of any mechanical system).

A3. The uncertain term

$$\xi(t) := \vartheta(t) - \tilde{C}\left(\delta(t), \dot{\delta}(t)\right)\dot{\delta}(t) - \tilde{G}(\delta(t)) \tag{20.6}$$

is admitted to be unknown and unmeasurable, but is bounded as

$$\|\xi(t)\| \leq c + c_0\,\|\delta(t)\| + c_1\left\|\dot{\delta}(t)\right\|, c, c_0, c_1 \geq 0. \tag{20.7}$$

A4. The loss function $F : \mathbb{R}^n \to \mathbb{R}^1$, characterizing the quality of a controlled process, is assumed to be **unknown**, convex (not obligatory, strongly convex), differentiable for almost all $\delta \in \mathbb{R}^n$ (the Radamacher theorem) and its *sub-gradient* $a(\delta)$ is supposed to be *measurable*[1] and bounded at any point δ_1, that is,

$$(\|a(\delta(t))\| \;\leq\; d_g \;<\infty)$$

and the reaction $a(\delta)$ is available for any argument $\delta \in R^n$.

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[1] By the definition (see (**?**)) a vector $a \in R^n$, satisfying the inequality

$$F(x+y) \geq F(x) + a^{\intercal}(x)y$$

for all $y \in R^n$, is called the sub-gradient of the function $F(x)$ at the point $x \in R^n$ and is denoted by $a(x) \in \partial F(x)$ - the set of all subgradients of $F(x)$ at the point x. If $F(x)$ is differentiable at a point x, then $a(x) = \nabla F(x)$. In the minimal point x^* we have $0 \in \partial F(x^*)$.

A5. The minimum of the loss function $F(\delta)$ exists, namely, [2]

$$F^* = \min_{\delta \in \mathbb{R}^n} F(\delta) > -\infty.$$

Problem 20.1 *Under the assumptions A1-A3 we need to design a control strategy $\tau(t)$ as a feedback $\tau(\delta(\cdot))$, which provides the **functional convergence** of the cost function $F(\delta(t))$ to its minimum value F^*, in the presence of uncertainties $\xi(t)$, that is, to guarantee*

$$\boxed{F(\delta(t)) \underset{t\longrightarrow\infty}{\longrightarrow} \inf_{\delta\in\mathbb{R}^n} F(\delta) = F^*,} \tag{20.8}$$

*supposing that the current **sub-gradient** $a(\delta(t))$ of the convex function $F(\delta)$, to be optimized, is available on-line.*

The convex (not obligatory strongly) loss function $F : \mathbb{R}^n \to \mathbb{R}^1$ defines the quality of control actions $\{\tau(t)\}_{t\geq 0}$ in the point $\delta(t)$. For example, the following two functions belong to the considered class of the convex loss functions to be optimized:

1.
$$F(\delta) = \sum_{i=1}^{n} |\delta_i|, a_i(\delta) = \operatorname{sign}(\delta_i),$$

2.
$$F(\delta) = \sum_{i=1}^{n} |\delta_i|_\varepsilon^+, \quad |z|_\varepsilon^+ := \begin{cases} z - \varepsilon & \text{if} \quad z \geq \varepsilon \\ -z - \varepsilon & \text{if} \quad z \leq -\varepsilon \\ 0 & \text{if} \quad |z| < \varepsilon \end{cases},$$

$$a_i(\delta_i) = \begin{cases} 1 & \text{if} \quad \delta_i \geq \varepsilon \\ -1 & \text{if} \quad \delta_i \leq -\varepsilon \\ (-1,1) & \text{if} \quad |\delta_i| < \varepsilon \end{cases} = \operatorname{sign}(|\delta| - \varepsilon).$$

In both these examples

$$F^* = F(0) = 0.$$

[2] In some problems the minimum of a loss function may be negative. For example, in conservative systems a stable equilibrium by the Lagrange-Dirichlet theorem corresponds to the minimum of potential energy which admits to have negative values.

20.0.3 Desired dynamics and its properties

Auxilary sliding variable $s(t)$

Define the vector function $s(t) \in \mathbb{R}^n$, which from now on and throughout this lecture will be referred to as *"sliding variable"*:

$$s(t) = \dot{\delta}(t) + \frac{\delta(t) + \eta}{t + \theta} + \tilde{G}(t), \ \eta = \text{const} \in \mathbb{R}^n,$$

$$\tilde{G}(t) := \frac{1}{t + \theta} \int_{\tau = t_0}^{t} a(\delta(\tau))\, d\tau, \ \theta > 0, \tag{20.9}$$

$$a(\delta_1(\tau)) \in \partial F(\delta_1(\tau))$$

Here $\delta(t) \in \mathbb{R}^n$ is defined in (20.2), η is a constant vector and $\tilde{G}(t)$ is the averaged subgradient (ASG) of the function $F(\delta(t))$ (23.39).

Remark 20.1 *Note that the sliding variable $s(t)$ contains the integral term which is physically measurable.*

Desired dynamic

Define the desired ASG dynamics as

$$s(t) = \dot{s}(t) = 0, \, t \geq t_0, \tag{20.10}$$

which corresponds exactly to the situation when the sliding variable $s(t)$ is equal to zero for all $t \geq t_0$. Below we will show why the dynamic (23.44) is called a *desired*. Since

$$\left.\begin{array}{c} (t + \theta)\, s(t) = (t + \theta)\, \dot{\delta}(t) + \delta(t) + \eta = \zeta(t), \\ \\ \dot{\zeta}(t) = -a(\delta(t)), \ \zeta(t_0) = 0, \end{array}\right\} \tag{20.11}$$

in the desired regime (23.42) we have

$$\boxed{\left.\begin{array}{c} (t + \theta)\, \dot{\delta}(t) + \delta(t) + \eta = \zeta(t), \quad t \geq t_0 \geq 0, \\ \\ t_0 \text{ is the moment when the desired dynamics may begin.} \end{array}\right\}} \tag{20.12}$$

Lemma 20.1 (Functional convergence in the desired regime.) *For the variable $\delta(t)$, satisfying the ideal dynamics (23.42), with any $\theta > 0$ and η, for all $t \geq t_0 \geq 0$ the following inequality is guaranteed:*

$$\boxed{F(\delta(t)) - F^* \leq \frac{\Phi(t_0)}{t+\theta} \underset{t\to\infty}{\to} 0,} \tag{20.13}$$

where

$$\Phi(t_0) = \Phi(\delta(t_0), \theta, \eta) := (t_0 + \theta) F(\delta(t_0)) - F^* + \frac{1}{2}\|\delta^* - \eta\|^2. \tag{20.14}$$

and

$$\begin{gathered} \delta^* \in \mathrm{Arg}_{\inf} \inf_{\delta \in \mathbb{R}^n} F(\delta) \\ (\delta^* \textit{ may be not unique}). \end{gathered} \tag{20.15}$$

Proof. Defining $\mu(t) := t + \theta$ we have

$$\frac{d}{dt}\left[\frac{1}{2}\|\zeta(t)\|^2 - \zeta^{\intercal}(t)\delta^*\right] = \dot{\zeta}^{\intercal}(t)(\zeta(t) - \delta^*) =$$

$$-a^{\intercal}(\delta(t))\left[\mu(t)\dot{\delta}(t) + \delta(t) + \eta - \delta^*\right] =$$

$$-a^{\intercal}(\delta(t))(\delta(t) - \delta^*) - a^{\intercal}(\delta(t))\left(\mu(t)\dot{\delta}(t) + \eta\right).$$

Using the inequality (see Chapter 23 in [7])

$$(\delta - \delta^*)^T a(\delta) \geq F(\delta) - F^*,$$

valid for convex (not obligatory stongly convex) functions in the first term on the right side, and applying the identity

$$a^T(\delta(t))\dot{\delta}(t) = \frac{d}{dt}[F(\delta(t)) - F^*],$$

we get

$$\frac{d}{dt}\left[\frac{1}{2}\|\zeta(t)\|^2 - \zeta^{\intercal}(t)\delta^*\right] \leq -[F(\delta(t)) - F^*]$$

$$-\mu(t)\frac{d}{dt}[F(\delta(t)) - F^*] - a^T(\delta(t))\eta.$$

Then, integrating the last inequality in the interval $[t_0, t]$ and applying the formula of integration by parts, we derive

$$\int_{\tau=t_0}^{t} [F(\delta(\tau)) - F^*]\, d\tau \leq \frac{1}{2}\left(\|\zeta(t_0)\|^2 - \|\zeta(t)\|^2\right) +$$

$$(\zeta(t) - \zeta(t_0))^T \delta^* - (\mu(t)[F(\delta(t)) - F^*])_{t_0}^{t} +$$

$$\int_{\tau=t_0}^{t} [F(\delta(\tau)) - F^*]\, \dot{\mu}(\tau)\, d\tau - \left[\int_{\tau=t_0}^{t} a^{\intercal}(\delta(\tau))\, d\tau\right]\eta.$$

Since $\dot{\mu}_\tau = 1$, the above inequality becomes

$$\left.\begin{array}{c}
\mu(t)[F(\delta(t)) - F^*] \leq \mu(t_0)[F(\delta(t_0)) - F^*] + \\
\frac{1}{2}\left(\|\zeta(t_0)\|^2 - \|\zeta(t)\|^2\right) + (\zeta(t) - \zeta(t_0))^{\intercal}\delta^* + \zeta^{\intercal}(t)\,\eta = \\
(t_0 + \theta)[F(\delta(t_0)) - F^*] + \left(\frac{1}{2}\|\zeta(t_0)\|^2 - \zeta^{\intercal}(t_0)\,\delta^*\right) + \\
\frac{1}{2}\|\delta^* - \eta\|^2 - \frac{1}{2}\underbrace{\left[\|\zeta(t)\|^2 - 2\,\zeta^{\intercal}(t)(\delta^* - \eta) + \|\delta^* - \eta\|^2\right]}_{\|\zeta(t)-(\delta^*-\eta)\|^2} \\
\leq (t_0 + \theta)[F(\delta(t_0)) - F^*] - \frac{1}{2}\|\zeta(t) - (\delta^* - \eta)\|^2 + \\
\left(\frac{1}{2}\|\zeta(t_0)\|^2 - \zeta^{\intercal}(t_0)\,\delta^*\right) + \frac{1}{2}\|\delta^* - \eta\|^2 \leq \Phi_{t_0},
\end{array}\right\} \qquad (20.16)$$

from which we obtain (23.46). Lemma is proved. ■

Remark 20.2 *The parameter η will be chosen below in such a way that the desired optimization regime starts from the beginning of the process, namely, when, $t_0 = 0$.*

Corollary 20.1 *In the partial case when*

$$\delta^* = 0,\ t_0 = 0 \text{ and } F^* = 0$$

the formula (23.46) becomes

$$\boxed{\Phi(t_0) = \Phi(\delta(t_0), \theta, \eta) := \theta F(\delta(0)) + \frac{1}{2}\|\eta\|^2.} \qquad (20.17)$$

20.0.4 Main theorem on ASG robust controller

Theorem 20.1 *Under assumptions 1-5 the ISM robust controller*

$$\left.\begin{array}{c} \tau(t) = \tilde{D}(\delta(t))\left[-k_t \mathrm{SIGN}(s(t)) + u_{comp}(t)\right], \\ u_{comp}(t) = -p_t^{reali}, \\ k_t = \left\|\tilde{D}^{-1}(\delta(t))\right\| \left(c + c_0\|\delta(t)\| + c_1\left\|\dot{\delta}(t)\right\|\right) + \rho_0, \ \rho_0 > 0, \end{array}\right\} \tag{20.18}$$

where

$$p_t^{reali} := \frac{1}{t+\theta}\left(\dot{\delta}(t) - \frac{\delta(t)+\eta}{t+\theta} - \tilde{G}(t) + a(\delta(t))\right) \tag{20.19}$$

with

$$\eta = -\theta\delta_{2,0} - \delta_{1,0} \tag{20.20}$$

guarantees the functional convergence (23.45) from the beginning of the process ($t_0 = 0$).

Proof. *In view of the assumption A2 we have that the matrix $D(q)$ is invertible, and then, by (20.5), it follows*

$$\left.\begin{array}{c} \delta(t) := q(t) - q^*(t), \ \dot{\delta}(t) = \dot{q}(t) - \dot{q}^*(t), \\ \ddot{\delta}(t) = \tilde{D}^{-1}(\delta(t))\tau(t) + \tilde{D}^{-1}(\delta(t))\xi(t). \end{array}\right\}$$

For the Lyapunov function $V(s) = \frac{1}{2}s^{\intercal}s$ we have

$$\begin{array}{c} \dot{V}(s(t)) = s^{\intercal}(t)\dot{s}(t) = \\ s^{\intercal}(t)\left(\ddot{\delta}(t) + \frac{\dot{\delta}(t)}{t+\theta} - \frac{\delta(t)+\eta}{(t+\theta)^2} - \frac{1}{t+\theta}\tilde{G}(t) + \frac{1}{t+\theta}a(\delta(t))\right) = \\ s^{\intercal}(t)\left(\tilde{D}^{-1}(\delta(t))\tau(t) + \tilde{D}^{-1}(\delta(t))\xi(t)\right) + \\ s^{\intercal}(t)\left(\frac{1}{t+\theta}\left(\dot{\delta}(t) - \frac{\delta(t)+\eta}{t+\theta} - \tilde{G}(t) + a(\delta(t))\right)\right) = \\ s^{\intercal}(t)p_t^{reali} + s^{\intercal}(t)\tilde{D}^{-1}(\delta(t))\tau(t) + s^{\intercal}(t)\tilde{D}^{-1}(\delta(t))\xi(t). \end{array} \tag{20.21}$$

Selecting τ as in (20.18) for the second term in (20.21) we get

$$\dot{V}(s_t) = -k_t s^{\intercal}(t)\,\mathrm{SIGN}(s(t)) + s^{\intercal}(t)\,\tilde{D}^{-1}(\delta(t))\,\xi(t)$$
$$\leq -k_t \sum_{i=1}^{n} |s_i(t)| + \|s(t)\| \left\|\tilde{D}^{-1}(\delta(t))\right\| \|\xi(t)\| \tag{20.22}$$

Taking into account that

$$\sum_{i=1}^{n} |s_i(t)| \geq \|s(t)\|$$

and, in view of (20.7) and (20.22), we derive

$$\begin{gathered}\dot{V}(s(t)) \leq -k_t \|s(t)\| + \\ \|s(t)\| \left\|\tilde{D}^{-1}(\delta(t))\right\| \left(c + c_0 \|\delta(t)\| + c_1 \left\|\dot{\delta}(t)\right\|\right) = \\ -\rho_0 \|s(t)\| = -\sqrt{2}\rho_0 \sqrt{V(s(t))},\end{gathered}$$

implying

$$2\left(\sqrt{V(s(t))} - \sqrt{V(s(t_0))}\right) \leq -\sqrt{2}\rho_0 t$$

and

$$0 \leq \sqrt{V(s(t))} \leq \sqrt{V(s(t_0))} - \frac{\rho_0}{\sqrt{2}} t,$$

which leads to the conclusion that for all

$$t \geq t_{reach} := \frac{1}{\rho_0}\sqrt{2V(s_{t_0})} = \frac{\|s_{t_0}\|}{\rho_0}$$

we have that $V(s(t)) = 0$ and $s(t) = 0$. To make the reaching time $t_{reach} = 0$ it is sufficient to gurantee that $s_{t_0=0} = 0$. But since by (23.43)

$$(t+\theta)\, s(t) = (t+\theta)\,\dot{\delta}(t) + \delta(t) + \eta = \zeta(t),$$

$$(t_0+\theta)\, s(t_0) = (t_0+\theta)\,\dot{\delta}(t_0) + \delta(t_0) + \eta = \zeta(t_0)$$

$$s_{t_0} = \dot{\delta}_{t_0} + \frac{\delta_{t_0} + \eta}{t_0 + \theta},$$

we need to fulfill the condition $s_{t_0=0} = 0:$

$$s_{t_0=0} = \dot{\delta}_{t_0=0} + \frac{\delta_{t_0=0} + \eta}{\theta} = 0,$$

which is possible if take η as in (23.48), providing

$$t_{reach} = \frac{\|s_0\|}{\rho_0} = 0.$$

Theorem is proven. ■

20.1 Exercise

Exercise 20.1 *For the system*

$$\left.\begin{array}{c}\frac{d^2}{dt^2}\bar{x}\,(t) = a_1\frac{\bar{x}\,(t)}{1+|\bar{x}\,(t)|} + a_2 \arctan\left(\frac{d}{dt}\bar{x}\,(t)\right) + \tau + d_0 \sin(\omega t),\\ \textit{with}\\ \bar{x}\,(0) = 1,\ \frac{d}{dt}\bar{x}\,(0) = 0,\ a_1 = -0.5,\ a_2 = 0.1,\ d_0 = 0.01,\ \omega = 10.\end{array}\right\}$$

as in Exercise 14.1, but assuming that $\bar{x}\,(t)$ *and* $\frac{d}{dt}\bar{x}\,(t)$ *are measurable (available on-line) and supposing that* d_0 *and* ω *are not known, i.e.,* $\vartheta\,(t) = d_0 \sin(\omega t)$*, design the control feedback* τ *which provides a good tracking for the process*

$$\left.\begin{array}{c}q^*(t) = A\cos(\Omega t),\\ \\ A = 2,\ \Omega = 0.1 - \ \textit{assumed to be known.}\end{array}\right\}$$