Lecture 19

Integral Sliding Mode

19.1 Main idea

In some control problem the control law, i.e. the nominal trajectory, is already done in the initial state space. The only the designers needed is to ensure the insensitivity of the trajectory tracking with respect uncertainties starting form the initial time moment. To ensure exact (with respect to the matched uncertainties/disturbances, acting in the same subspace as control) tracking of the nominal trajectory designed for nominal systems in original state space starting from initial time moment the concept of *integral sliding mode control* (ISMC) (see [8], [9]) were proposed (see also [10] where the application of ISM to LQ problem and specific algebraic observers may be found).

The integral sliding surface is a surface in extended state space. The motions on this surface are starting from the initial time moment. So the systems governed by ISMC has the following advantages:

- compensation of the matched uncertainties/ disturbances is starting from initial time moment since the motion surface is a virtual surface;
- the motions in integral sliding modes has a dimension of the initial state space;
- it leads to chattering reduction, because ISMC needs the smaller discontinuous control gains since the nominal systems dynamics supposed to be already compensated by nominal control law.

Unfortunately the main drawbacks of ISMC are:

- they need a complete information about all of system´s states starting from initial time moment;
- ISMC can not compensate unmatched uncertainties.

To illustrate the main idea of ISMC consider now the following simplest uncertain model:

$$\dot{x}_1(t) = x_2(t),\ \dot{x}_2(t) = u(t) - f(t,x), \tag{19.1}$$

where the uncertain term $f(t,x)$ is assumed to be bounded as

$$|f(t,x)| \leq L$$

Assume also that the desired dynamics

$$\dot{x}_1^*(t) = x_2^*(t),\ \dot{x}_2^*(t) = u_0(t) \tag{19.2}$$

is obtained by the application of the controo law $u_0(t)$ which is supposed to be known. The initial condition of both dynamics are supposed to be equal. Select

$$u(t) = u_0(t) + u_1(t).$$

Define auxilary scalar variable s as follows

$$s(t) = \sigma(t) + x_2(t). \tag{19.3}$$

The function σ we will select below. Then we have

$$\dot{s}(t) = \dot{\sigma}(t) + \dot{x}_2(t) = \dot{\sigma}(t) + (u(t) - f(t,x)) =$$

$$\dot{\sigma}(t) + [u_0(t) + u_1(t)] - f(t,x).$$

Design $u_1(t)$ as

$$u_1(t) = -k\text{sign}\,(s(t))\,,$$

which leads to

$$\dot{s}(t) = \dot{\sigma}(t) + u_0(t) - k\text{sign}\,(s(t)) - f(t,x),$$

and hence, for

$$V\,(s) = \frac{1}{2}s^2$$

we get

$$\dot{V}(s(t)) = s(t)\dot{s}(t) = s(t)\left[\dot{\sigma}(t) + u_0(t) - k\text{sign}(s(t)) - f(t,x)\right] =$$

$$-k|s(t)| + s(t)\left[\dot{\sigma}(t) + u_0(t) - f(t,x)\right].$$

Select now

$$\dot{\sigma}(t) := -u_0(t),\ \sigma(0) = -x_2(0),$$

implying (see (19.3))

$$s(0) = \sigma(0) + x_2(0) = 0.$$

Then we get

$$\dot{V}(s(t)) = -k|s(t)| - s(t)f(t,x) \le -k|s(t)| + |s(t)||f(t,x)| \le$$

$$-(k-L)|s(t)| = -(k-L)\sqrt{2V(s(t))},$$

which for $k > L$ leads to the following result:

$$V((s(t))) = 0 \text{ for all } t \ge t_{reach} = \frac{|s(0)|}{k-L} = 0.$$

Now we are ready to formulate the following result.

Lemma 19.1 *The control law $u(t)$, referred to as the* ***integral sliding mode control*** *(because it contains the integral term) of the form*

$$u(t) = u_0(t) - k\text{sign}(s(t)),$$

$$s(t) = \sigma(0) - \int_{\tau=0}^{t} u_0(\tau)\,d\tau + x_2(t) =$$

$$x_2(t) - x_2(0) - \int_{\tau=0}^{t} u_0(\tau)\,d\tau$$

maintains the property

$$s(t) = 0$$

from the begginning of the process, that is, for all $t \ge 0$ implying

$$0 = \dot{s}(t) = \dot{\sigma}(t) + (u_0(t) + u_1(t)) - f(t,x) = u_1 - f(t,x)$$

or, equivalently, $u_{1,eq}(t) = f(t,x)$, *maintaining the dynamics*

$$\frac{d}{dt}: \ s(t) = \sigma(0) - \int_{\tau=0}^{t} u_0(\tau)\, d\tau + x_2(t) = x_2(t) - x_2(0) - \int_{\tau=0}^{t} u_0(\tau)\, d\tau$$

$$\Downarrow$$

$$\dot{x}_1(t) = x_2(t), \ \dot{x}_2(t) = u_0(t),$$

which for the equal initial conditions

$$x_1(0) = x_1^*(0), \ x_2(0) = x_2^*(0)$$

provides for the uncertain system (19.1) the desired dynamics

$$x_1(t) = x_1^*(t), x_2(t) = x_2^*(t)$$

(19.2) from ***the begging of the process****, i.e., for all* $t \geq 0$.

19.2 Problem Formulation in general affine format

Consider now the following controlled uncertain system represented by the state-space equation

$$\boxed{\left.\begin{array}{c} \dot{x}(t) = f(x(t)) + B(x(t))\, u(t) + \phi(x,t), \\ \\ x(0) \ \text{ is given,} \end{array}\right\}} \tag{19.4}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector. The function $\phi(x,t)$ represents the uncertainties affecting the system due to parameter variations, unmodelled dynamics and/or exogenous disturbances. Let $u(t) = u_0(t)$ be a nominal control designed for (19.4) assuming $\phi(x,t) = 0$, where $u_0(t)$ is designed to achieve a desired task, whether it be stabilization, tracking or an optimal control problem. Thus, the trajectories $x_0(t)$ of the ideal system ($\phi(x,t) = 0$) will be given by the solutions of the following ODE equations,

$$\boxed{\dot{x}_0(t) = f(x_0(t)) + B(x_0(t))\, u_0(t).} \tag{19.5}$$

For

$$x(0) = x_0(0)$$

and $\phi(x,t)$ being not equal to zero, the trajectories of (19.4) and (19.5) are different. The trajectories of (19.5) satisfy some specified requirements, whereas the trajectories of (19.5) might have a quite different performance (depending on $\phi(x,t)$) to the one expected by the control designer.
To realize the control design given below we need to assume that:

1) for all $x \in \mathbb{R}^n$

$$\operatorname{rank} B(x) = m,$$

that is,

$$\boxed{B^{\mathsf{T}}(x) B(x) > 0;}$$

2) the disturbance $\phi(x,t)$ is assumed to be **matched**, i.e., it satisfies the, so-called, *matching condition*

$$\phi(x,t) \in \operatorname{Im} B(x),$$

i.e., there exists a vector $\gamma(x,t) \in \mathbb{R}^m$ such that

$$\boxed{\phi(x,t) = B(x)\gamma(x,t).} \tag{19.6}$$

From a control point of view, the matching condition means that the effects, produced by $\phi(x,t)$ in the system, can be produced by $u(t)$, and vice versa;

3) an upper bound $\gamma^{+}(x,t)$ for $\gamma(x,t)$ exists and is known, i.e.,

$$\boxed{\|\gamma(x,t)\| \leq \gamma^{+}(x,t).} \tag{19.7}$$

Obviously, the second restriction is needed to compensate $\phi(x,t)$; if it is known, it would be enough to chose $u(t) = -\gamma(x,t)$. However, since $\gamma(x,t)$ is uncertain, some other restrictions are needed in order to eliminate the influence of $\phi(x,t)$. In this way, the sliding mode approach replaces the lack of knowledge of $\phi(x,t)$ by the first and third assumptions.

19.2.1 Control Design Objective

Now the control design problem is *to design a control law* that, provided that

$$x(0) = x_0(0),$$

guarantees the identity

$$x(t) = x_0(t)$$

for all $t \geq 0$. By comparing (19.4) and (19.5), it is clear that the control design is achieved only if the equivalent control is equal to the negative of the uncertainty ($u_{1_{\text{eq}}}(t) = -\gamma(x,t)$). Thus, the control objective can be reformulated in the following terms: *design the control $u = u(t)$ in the following form*

$$\boxed{u(t) = u_0(t) + u_1(t),} \tag{19.8}$$

where $u_0(t)$ is the nominal control part designed for (19.5) and $u_1(t)$ is the *integral sliding mode* (ISM) control part guarantying the compensation of the **unmeasured matched uncertainty** $\phi(x,t)$, starting from the beginning $(t = 0)$ of the process.

19.2.2 ISM Control Design

Since $\phi(x,t) = B(x)\gamma(x,t)$, substitution of (19.8) into (19.4) yields

$$\dot{x}(t) = f(x(t)) + B(x(t))(u_0(t) + u_1(t) + \gamma(x,t)). \tag{19.9}$$

Define $s(t)$ as

$$\boxed{\begin{array}{c} s(t) = s_0(x(t)) - s_0(x(0)) - \\ \int_0^t G(x(\tau))[f(x(\tau)) + B(x(\tau))u_0(\tau)]\,d\tau, \\ \\ s_0(x) \in \mathbb{R}^m,\ G(x(t)) = \dfrac{\partial s_0}{\partial x}(t), \end{array}} \tag{19.10}$$

where $s_0(x) \in \mathbb{R}^m$ is any vector function satisfying

$$\det\left[\frac{\partial s_0(x)}{\partial x}B(x)\right] \neq 0 \text{ for all } x \in \mathbb{R}^n.$$

Then

$$\begin{gathered} \dot{s}(x(t)) = \dot{s}_0(x(t)) - G(x(t))[f(x(t)) + B(x(t))u_0(t)] = \\ G(x(t))\dot{x}(t) - G(x(t))[f(x(t)) + B(x(t))u_0(t)] = \\ G(x(t))[f(x) + B(x)(u_0(t) + u_1(t) + \gamma(x,t))] - \\ G(x(t))[f(x(t)) + B(x(t))u_0(t)] = G(x)B(x)(u_1(t) + \gamma(x,t)). \end{gathered}$$

In the contrast with conventional sliding modes, here an integral term is included. Furthermore, in this case for any function $s_0(x)$ (19.10) we have

$$s(x(0)) = 0.$$

Let us design the sliding mode control as

$$\left.\begin{array}{c} u_1(t) = -M(x(t),t)\,\dfrac{D^{\mathsf{T}}(x(t))\,s(t)}{\|D^{\mathsf{T}}(x(t))\,s(t)\|}, \\ \\ M(x(t),t) > \gamma^{+}(x(t),t), \;\; D(x(t)) := G(x(t))\,B(x(t)). \end{array}\right\} \tag{19.11}$$

Taking $V(s) = \dfrac{1}{2}s^T s$, and in view of (19.7) the time derivative of $V(s)$ is bounded as follows

$$\dot{V}(t) = (s(t), \dot{s}(t)) = (s(t), D(x(t))(u_1(t) + \gamma(x(t),t))) =$$

$$(D^{\mathsf{T}}(x(t))\,s(t), u_1(t) + \gamma(x(t),t)) \leq$$

$$-\|D^{\mathsf{T}}(x(t))\,s(t)\|\,(M(x(t),t) - \gamma^{+}(x(t),t)) < 0$$

Hence $V\ (s)$decreases, which implies

$$V(t) \leq V(0) = \frac{1}{2}\|s(x(0))\|^2 = 0.$$

That is, **the sliding mode is achieved from the beginning**. Now, the equivalent control $u_{1_{\text{eq}}}$ is taken from

$$\dot{s} = D(x(t))\,(u_1(t) + \gamma(x(t),t)) = 0 \Rightarrow u_1(t) + \gamma(x(t),t) = 0.$$

Thus, in this case,

$$u_{1_{\text{eq}}}(t) = -\gamma(x(t),t).$$

Hence,by (19.9) the sliding motion is given by

$$\dot{x}(t) = f(x(t)) + B(x(t))\,u_0(t),$$

and our aim is achieved since now

$$x(t) \equiv x_0(t).$$

Notice that the order of the dynamic equation in the sliding mode is not reduced. This property defines an *integral sliding mode* [10].

19.3 Exercises

Exercise 19.1 *For the system*

$$\left.\begin{array}{c} \ddot{x}(t) = -\omega^2 x(t) + u(t) + \phi(x,t) \\ \text{with} \\ \phi(x,t) = 0.1\arctan(\dot{x}(t)) + 0.01\sin(10t), \\ \omega = 0.5,\ x(0) = -1,\ \dot{x}(0) = 1, \end{array}\right\}$$

design the control $u(t)$, using ISM method, such that from the beggining of the process $t = 0$ the trajectory of the controlled system would conside with the desired trajectory $x^*(t)$, generated by the dynamic model

$$\left.\begin{array}{c} \ddot{x}^*(t) = -\omega_0^2 x^*(t) \\ \text{with the same initial conditions} \\ x^*(0) = -1,\ \dot{x}^*(0) = 1 \\ \text{and} \\ \omega_0 = 0.3. \end{array}\right\}$$

Hint. *Represent the given system es*

$$\left.\begin{array}{c} \dot{x}_1(t) = x_2(t) \\ \\ \dot{x}_2(t) = -\omega^2 x_1(t) + u(t) + \phi(x,t) \end{array}\right\}$$

$$\Updownarrow$$

$$x_1(t) = x(t),\ x_2(t) = \dot{x}(t) = \dot{x}_1(t),$$

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\mathbf{x} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u + \begin{pmatrix} 0 \\ \phi(\mathbf{x},t) \end{pmatrix}$$

Select

$$\left.\begin{array}{c} u(t) = u_0(t) + u_1(t) \\ \text{with} \\ u_0(t) = \left(\omega^2 - \omega_0^2\right) x_1(t), \end{array}\right\}$$

and

$$s_0(\mathbf{x}) = B^{\intercal}\mathbf{x}.$$

Exercise 19.2 *For the same system as in Exercise (17.1), supposing that*

$x(t)$ and $\dot{x}(t)$ are available, and the desired dynamics satisfies

$$\left.\begin{array}{c}\ddot{x}^*(t) + 2\dot{x}^*(t) + 5x^*(t) = u_0(t), \\ \textit{where } x^*(0), \dot{x}^*(0) \textit{ are given,} \\ \\ u_0(t) = u_0 \sin\left(\omega_0 t + \dfrac{3}{4}\pi\right)\end{array}\right\}$$

to construct the Integral Sliding Mode Controller.