Lecture 13

Dynamic Regulator

13.1 Full order linear dynamic controller

The suggested minimization strategy for quasi-Lipschitz nonlinear output control systems is combined with full-order linear dynamic output controllers in this lecture. In general, we select the controller parameters (gain matrices) that minimize the size of an appealing ellipsoid associated with the closed-loop control system.

Consider the quasi-Lipschitz nonlinear output control system (11.3), (11.4):

$$\left.\boxed{\begin{array}{c} \dot{x}(t) = Ax(t) + Bu(t) + \xi_x(t,x), \\ \\ x(0) = x_0 \in \mathbb{R}^n, \\ \\ y(t) = Cx(t) + \xi_y(t,x). \end{array}}\right\} \tag{13.1}$$

The admissible feedback control strategies for this system are chosen from the class of *full order linear dynamic controllers* of the following structure

$$\left.\boxed{\begin{array}{c} u(t) = C_r x_r(t) + D_r y(t), \\ \\ \dot{x}_r(t) = A_r x_r(t) + B_r y(t), \\ \\ x_r(0) = x_0^r. \end{array}}\right\} \tag{13.2}$$

where

$$x_r \in \mathbb{R}^n,\ A_r \in \mathbb{R}^{n\times n},\ B_r \in \mathbb{R}^{n\times k},\ D_r \in \mathbb{R}^{m\times k},\ C_r \in \mathbb{R}^{m\times n}$$

The control design associated with (13.2) is completely determined by a selection of the matrix

$$\boxed{\Theta := \begin{bmatrix} D_r & C_r \\ B_r & A_r \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+k)}} \tag{13.3}$$

We call Θ the *dynamic controller matrix*. The closed-loop realization of (13.1) under (13.2) is given by

$$\boxed{\left.\begin{array}{c} \dot{z} = (A_0 + B_0\Theta C_0)z + D_0\xi_x + B_0\Theta E_0\xi_y, \\ \\ z(t) := \begin{pmatrix} x(t) \\ x_r(t) \end{pmatrix} \in \mathbb{R}^{2n},\ z(0) = \begin{pmatrix} x_0 \\ x_{r,0} \end{pmatrix}, \end{array}\right\}} \tag{13.4}$$

where

$$A_0 := \begin{bmatrix} A & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix},\ B_0 := \begin{bmatrix} B & 0_{n\times n} \\ 0_{n\times m} & I_{n\times n} \end{bmatrix}$$
$$C_0 := \begin{bmatrix} C & 0_{k\times k} \\ 0_{n\times n} & I_{n\times n} \end{bmatrix},\ D_0 := \begin{bmatrix} I_{n\times n} \\ 0_{n\times n} \end{bmatrix},\ E_0 := \begin{bmatrix} I_{n\times k} \\ 0_{k\times k} \end{bmatrix}.$$

As in the previouse Chepter 12 define the matrix $\tilde{G}_B$ which satisfies the relations

$$\tilde{G}_B\tilde{G}_B^T = \tilde{G}_B^T\tilde{G}_B = I_{2n\times 2n}\,,\ \tilde{G}_B B_0 = \begin{bmatrix} \left[0_{(2n-\tilde{m})\times\tilde{m}}\right] \\ \tilde{B}_0 \end{bmatrix} \in \mathbb{R}^{2n\times\tilde{m}},$$

$$\tilde{B}_0 \in \mathbb{R}^{\tilde{m}\times\tilde{m}},\ \det\tilde{B}_0 \neq 0$$

$$\tilde{G}_B = \begin{pmatrix} B_0^{\perp} \\ B_0' \end{pmatrix},\ \text{where } B_0^{\perp} = (\text{null}(B_0^{\intercal}))^{\intercal} \text{ and } B_0' = \left(\text{null}\left(B_0^{\perp}\right)\right)^{\intercal}.$$

Following then the same scheme of the analysis as before, we obtain

$$\frac{d}{dt}\left(\tilde{G}_B z\right) = (\tilde{G}_B A_0 + \tilde{G}_B B_0\Theta C_0)z + \tilde{G}_B D_0\xi_x + \tilde{G}_B B_0\Theta E_0\xi_y\,,$$

or, in new coordinates

$$\tilde{z} := \tilde{G}_B z = \begin{pmatrix} \tilde{z}_1 \in \mathbb{R}^{2n-\tilde{m}} \\ \tilde{z}_2 \in \mathbb{R}^{\tilde{m}} \end{pmatrix}$$

we get

$$\boxed{\begin{aligned}\frac{d}{dt}\tilde{z} = (\tilde{A}_0 + \begin{bmatrix} \left[0_{(2n-\tilde{m})\times(n+k)}\right] \\ Y \end{bmatrix} C_0 \tilde{G}_B^{\intercal})\tilde{z} \\ + \tilde{G}_B D_0 \xi_x + \begin{bmatrix} \left[0_{(2n-\tilde{m})\times(n+k)}\right] \\ Y \end{bmatrix} E_0 \xi_y,\end{aligned}} \tag{13.5}$$

where

$$\tilde{A}_0 := \tilde{G}_B A_0 \tilde{G}_B^{\intercal}$$

and

$$\boxed{Y := \tilde{B}_0 \Theta \in \mathbb{R}^{\tilde{m}\times(n+k)}.} \tag{13.6}$$

13.2 Main result on the attractive ellipsoid for a dynamic controller

Now we are ready to formulate the principle result concerning the attractive ellipsoid guaranteed by the dynamic controller (13.2).

Theorem 13.1 *If the matrices*

$$P_1 \in \mathbb{R}^{(2n-\tilde{m})\times(n-\tilde{m})},\ P_2 \in \mathbb{R}^{\tilde{m}\times\tilde{m}},\ Y \in \mathbb{R}^{\tilde{m}\times(n+k)}$$

and positive numbers $\alpha, \varepsilon > 0$ satisfy the following matrix inequalities

$$0 > W_{\alpha,\varepsilon} =$$

$$\begin{bmatrix} \begin{matrix} P_1 B_0^{\perp} A \left(B_0^{\perp}\right)^{\intercal} + \\ \left[P_1 B_0^{\perp} A \left(B_0^{\perp}\right)^{\intercal}\right]^{\intercal} \\ +\alpha P_1 \\ +\varepsilon\left(\tilde{Q}_x + \tilde{Q}_y\right) \end{matrix} & \begin{matrix} P_1 B_0^{\perp} A \left(B_0'\right)^{\intercal} + \\ \left[P_2 B_0' A \left(B_0^{\perp}\right)^{\intercal}\right]^{\intercal} \\ + \left[YC\left(B_0^{\perp}\right)^{\intercal}\right]^{\intercal} \end{matrix} & P_1 B_0^{\perp} & 0_{(2n-\tilde{m})\times k} \\ \begin{matrix} \left[P_1 B_0^{\perp} A \left(B_0'\right)^{\intercal}\right]^{\intercal} \\ +P_2 B_0' A \left(B_0^{\perp}\right)^{\intercal} \\ +YC\left(B_0^{\perp}\right)^{\intercal} \end{matrix} & \begin{matrix} P_2 B_0' A \left(B_0'\right)^{\intercal} + \\ \left[P_2 B_0' A \left(B_0'\right)^{\intercal}\right]^{\intercal} \\ + \left[YC\left(B_0'\right)^{\intercal}\right]^{\intercal} + \\ YC\left(B_0'\right)^{\intercal} + \alpha P_2 \end{matrix} & 0_{\tilde{m}\times n} & \begin{matrix} P_2 B_0' \\ +P_2 Y \end{matrix} \\ \left(P_1 B_0^{\perp}\right)^{\intercal} & 0_{n\times\tilde{m}} & -\varepsilon I_{n\times n} & 0_{n\times k} \\ 0_{(2n-\tilde{m})\times k} & \left(P_2 B_0'\right)^{\intercal} + \left(P_2 Y\right)^{\intercal} & 0_{k\times n} & -\varepsilon I_{k\times k} \end{bmatrix}, \tag{13.7}$$

then the ellipsoid

$$\mathcal{E}_0(\tilde{P}_{attr}) := \{\tilde{z} \in \mathbb{R}^{2n} : \tilde{z}^T \tilde{P}_{attr} \tilde{z} < 1\},\ \tilde{P} = \begin{bmatrix} P_1 & 0_{(2n-\tilde{m})\times\tilde{m}} \\ 0_{\tilde{m}\times(2n-\tilde{m})} & P_2 \end{bmatrix}$$

is attractive for the system (13.5) with

$$\tilde{P}_{attr} = \frac{\alpha}{\varepsilon\left(c_{0,x} + c_{0,y}\right)} \tilde{P}$$

and the control matrix Θ *given by*

$$\Theta = \tilde{B}_0^{-1} Y.$$

Proof. Consider the quadratic function of the form

$$V(\tilde{z}) = \tilde{z}^\intercal \tilde{P} \tilde{z},\ \tilde{P} = \begin{bmatrix} P_1 & 0_{(2n-\tilde{m})\times\tilde{m}} \\ 0_{\tilde{m}\times(2n-\tilde{m})} & P_2 \end{bmatrix},$$

$$P_1 \in \mathbb{R}^{(2n-\tilde{m})\times(2n-\tilde{m})},\ P_2 \in \mathbb{R}^{\tilde{m}\times\tilde{m}},$$

and calculate its total derivative along the trajectories of the system (13.5):

$$\dot{V}(\tilde{z}) = 2\tilde{z}^\intercal \tilde{P} \frac{d}{dt}\tilde{z} = \tilde{z}^\intercal \left(\tilde{P}\tilde{A}_0 + \tilde{A}_0^\intercal \tilde{P}^\intercal\right)\tilde{z} +$$

$$\tilde{z}^\intercal \left(\begin{bmatrix} 0_{(2n-\tilde{m})\times(n+k)} \\ Y \end{bmatrix} C_0 \tilde{G}_B^\intercal + \tilde{G}_B C_0^\intercal \begin{bmatrix} 0_{(2n-\tilde{m})\times(n+k)} & Y^\intercal \end{bmatrix} \right) \tilde{z} \quad (13.8)$$

$$+ 2\tilde{z}^\intercal \tilde{P} \tilde{G}_B D_0 \xi_x + 2\tilde{z}^\intercal \tilde{P} \begin{bmatrix} 0_{(2n-\tilde{m})\times\times(n+k)} \\ Y \end{bmatrix} E_0 \xi_y$$

Taking into account that

$$\tilde{P}\tilde{A}_0 := \tilde{P}\tilde{G}_B A_0 \tilde{G}_B^\intercal =$$

$$\begin{bmatrix} P_1 & 0_{(2n-\tilde{m})\times\tilde{m}} \\ 0_{\tilde{m}\times(2n-\tilde{m})} & P_2 \end{bmatrix} \begin{pmatrix} B_0^\perp \\ B_0' \end{pmatrix} \begin{bmatrix} A & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} \begin{pmatrix} B_0^\perp \\ B_0' \end{pmatrix}^\intercal =$$

$$\begin{bmatrix} P_1 & 0_{(2n-\tilde{m})\times\tilde{m}} \\ 0_{\tilde{m}\times(2n-\tilde{m})} & P_2 \end{bmatrix} \begin{bmatrix} B_0^\perp A & 0_{n\times n} \\ B_0' A & 0_{n\times n} \end{bmatrix} \begin{pmatrix} \left(B_0^\perp\right)^\intercal & \left(B_0'\right)^\intercal \end{pmatrix} =$$

$$\begin{bmatrix} P_1 & 0_{(2n-\tilde{m})\times\tilde{m}} \\ 0_{\tilde{m}\times(2n-\tilde{m})} & P_2 \end{bmatrix} \begin{bmatrix} B_0^\perp A \left(B_0^\perp\right)^\intercal & B_0^\perp A \left(B_0'\right)^\intercal \\ B_0' A \left(B_0^\perp\right)^\intercal & B_0' A \left(B_0'\right)^\intercal \end{bmatrix} =$$

$$\begin{bmatrix} P_1 B_0^\perp A \left(B_0^\perp\right)^\intercal & P_1 B_0^\perp A \left(B_0'\right)^\intercal \\ P_2 B_0' A \left(B_0^\perp\right)^\intercal & P_2 B_0' A \left(B_0'\right)^\intercal \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} 0_{(2n-\tilde{m})\times(n+k)} \end{bmatrix} \\ Y \end{bmatrix} C_0 \tilde{G}_B^\intercal = \begin{bmatrix} \begin{bmatrix} 0_{(2n-\tilde{m})\times(n+k)} \end{bmatrix} \\ Y \end{bmatrix} \begin{bmatrix} C & 0_{k\times k} \\ 0_{n\times n} & I_{n\times n} \end{bmatrix} \begin{pmatrix} \left(B_0^\perp\right)^\intercal & \left(B_0'\right)^\intercal \end{pmatrix} = \begin{bmatrix} \begin{bmatrix} 0_{(2n-\tilde{m})\times(n+k)} \end{bmatrix} \\ Y \end{bmatrix} \begin{bmatrix} C\left(B_0^\perp\right)^\intercal & C\left(B_0'\right)^\intercal \\ 0_{n\times n} & \left(B_0'\right)^\intercal \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ YC\left(B_0^\perp\right)^\intercal & YC\left(B_0'\right)^\intercal \end{bmatrix},$$

$$\tilde{P}\tilde{G}_B D_0 = \begin{bmatrix} P_1 & 0_{(2n-\tilde{m})\times\tilde{m}} \\ 0_{\tilde{m}\times(2n-\tilde{m})} & P_2 \end{bmatrix} \begin{pmatrix} B_0^\perp \\ B_0' \end{pmatrix} \begin{bmatrix} I_{n\times n} \\ 0_{n\times n} \end{bmatrix} = \begin{bmatrix} P_1 B_0^\perp & 0_{(2n-\tilde{m})\times\tilde{m}} \\ 0_{\tilde{m}\times(2n-\tilde{m})} & P_2 B_0' \end{bmatrix}$$

and

$$\tilde{P} \begin{bmatrix} 0_{(2n-\tilde{m})\times(n+k)} \\ Y \end{bmatrix} E_0 = \begin{bmatrix} 0_{(2n-\tilde{m})\times} & 0_{(2n-\tilde{m})\times k} \\ 0_{\tilde{m}\times k} & P_2 Y \end{bmatrix},$$

we are able to represent (13.8) as

$$\dot{V}(\tilde{z}) = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \xi_x \\ \xi_y \end{pmatrix}^\intercal W_0 \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \xi_x \\ \xi_y \end{pmatrix}$$

with

$$W_0 = \begin{bmatrix} \begin{matrix} P_1 B_0^\perp A\left(B_0^\perp\right)^\intercal + \\ \left[P_1 B_0^\perp A\left(B_0^\perp\right)^\intercal\right]^\intercal \end{matrix} & \begin{matrix} P_1 B_0^\perp A\left(B_0'\right)^\intercal + \\ \left[P_2 B_0' A\left(B_0^\perp\right)^\intercal\right]^\intercal \\ + \left[YC\left(B_0^\perp\right)^\intercal\right]^\intercal \end{matrix} & P_1 B_0^\perp & 0_{(2n-\tilde{m})\times k} \\ \begin{matrix} \left[P_1 B_0^\perp A\left(B_0'\right)^\intercal\right]^\intercal \\ + P_2 B_0' A\left(B_0^\perp\right)^\intercal \\ + YC\left(B_0^\perp\right)^\intercal \end{matrix} & \begin{matrix} P_2 B_0' A\left(B_0'\right)^\intercal + \\ \left[P_2 B_0' A\left(B_0'\right)^\intercal\right]^\intercal + \\ YC\left(B_0'\right)^\intercal + \\ \left[YC\left(B_0'\right)^\intercal\right]^\intercal \end{matrix} & 0_{\tilde{m}\times n} & \begin{matrix} P_2 B_0' + \\ P_2 Y \end{matrix} \\ \left(P_1 B_0^\perp\right)^\intercal & 0_{n\times\tilde{m}} & 0_{n\times n} & 0_{n\times k} \\ 0_{k\times(2n-\tilde{m})} & \begin{matrix} \left(P_2 B_0'\right)^\intercal + \\ \left(P_2 Y\right)^\intercal \end{matrix} & 0_{k\times n} & 0_{k\times k} \end{bmatrix}$$

Adding and subtracting the term $-\alpha V(\tilde{z})$ and in the diagonal blocks $-\varepsilon Q_x$ and $-\varepsilon Q_y$ we derive

$$\dot{V}(\tilde{z}) = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \xi_x \\ \xi_y \end{pmatrix}^{\intercal} W_1 \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \xi_x \\ \xi_y \end{pmatrix} - \alpha V(\tilde{z}) + \varepsilon \left(\xi_x^{\intercal} \xi_x + \xi_y^{\intercal} \xi_y \right) \tag{13.9}$$

with

$$W_1 = \begin{bmatrix} \begin{matrix} P_1 B_0^{\perp} A \left(B_0^{\perp} \right)^{\intercal} + \\ \left[P_1 B_0^{\perp} A \left(B_0^{\perp} \right)^{\intercal} \right]^{\intercal} \\ +\alpha P_1 \end{matrix} & \begin{matrix} P_1 B_0^{\perp} A \left(B_0' \right)^{\intercal} + \\ \left[P_2 B_0' A \left(B_0^{\perp} \right)^{\intercal} \right]^{\intercal} \\ + \left[Y C \left(B_0^{\perp} \right)^{\intercal} \right]^{\intercal} \end{matrix} & P_1 B_0^{\perp} & 0_{(2n-\tilde{m}) \times k} \\ \begin{matrix} \left[P_1 B_0^{\perp} A \left(B_0' \right)^{\intercal} \right]^{\intercal} \\ +P_2 B_0' A \left(B_0^{\perp} \right)^{\intercal} \\ +Y C \left(B_0^{\perp} \right)^{\intercal} \end{matrix} & \begin{matrix} P_2 B_0' A \left(B_0' \right)^{\intercal} + \\ \left[P_2 B_0' A \left(B_0' \right)^{\intercal} \right]^{\intercal} \\ + \left[Y C \left(B_0' \right)^{\intercal} \right]^{\intercal} + \\ Y C \left(B_0' \right)^{\intercal} + \alpha P_2 \end{matrix} & 0_{\tilde{m} \times n} & P_2 B_0' + P_2 Y \\ \left(P_1 B_0^{\perp} \right)^{\intercal} & 0_{n \times \tilde{m}} & -\varepsilon I_{n \times n} & 0_{n \times k} \\ 0_{k \times (2n-\tilde{m})} & \left(P_2 B_0' \right)^{\intercal} + \left(P_2 Y \right)^{\intercal} & 0_{k \times n} & -\varepsilon I_{k \times k} \end{bmatrix}.$$

As in (11.15)

$$\begin{gathered} \varepsilon \left(\xi_x^{\intercal} \xi_x + \xi_y^{\intercal} \xi_y \right) \leq \varepsilon \left[\left(c_{0,x} + c_{0,y} \right) + \tilde{z}_1^{\intercal} \left(\tilde{Q}_x + \tilde{Q}_y \right) \tilde{z}_1 \right] \\ \tilde{Q}_x = \tilde{G}_B Q_x \tilde{G}_B^{\intercal}, \ \tilde{Q}_x = \tilde{G}_B Q_y \tilde{G}_B^{\intercal} \end{gathered}$$

In view of this, the ODE (13.9) can be converted into the following differential inequality

$$\dot{V}(\tilde{z}) \leq \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \xi_x \\ \xi_y \end{pmatrix}^{\intercal} W_{\alpha,\varepsilon} \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \xi_x \\ \xi_y \end{pmatrix} - \alpha V(\tilde{z}) + \varepsilon \left(c_{0,x} + c_{0,y} \right) \tag{13.10}$$

where $W_{\alpha,\varepsilon}$ is defined in (13.7). If $W_{\alpha,\varepsilon} < 0$, the result of this theorem follows from the considerations as in the proof of Theorems in the previouse lectures. ■

13.3 Optimal parameters of dynamic controller

Notice that matrix $W_{\alpha,\varepsilon}$ contains the term P_2Y, which makes the matrix inequality (13.7) nonlinear one. To deal with LMI constraint recall that the term corresponds to the term $2\tilde{z}_2^{\intercal}P_2Y\xi_x$ in the quadratic form representation. Using Λ-inequality (12.16) with $\Lambda = I_{n\times n}$, we get

$$2\tilde{z}_2^{\intercal}P_2Y\xi_x \leq \tilde{z}_2^{\intercal}\left(P_2\Lambda P_2^{\intercal}\right)\tilde{z}_2 + \xi_x^{\intercal}\left(Y^{\intercal}\Lambda^{-1}Y\right)\xi_x \leq$$

$$\tilde{z}_2^{\intercal}\left(P_2^{\intercal}P_2\right)\tilde{z}_2 + \xi_x^{\intercal}Y^{\intercal}Y\xi_x \leq \tilde{z}_2^{\intercal}Q_{\tilde{z}_2}\tilde{z}_2 + \xi_x^{\intercal}Q_{\xi_x}\xi_x$$

where matrices $Q_{\tilde{z}_2}$ and Q_{ξ_x} satisfy the matrix inequalities

$$P_2^{\intercal}P_2 < Q_{\tilde{z}_2},\ Y^{\intercal}P_2Y < Q_{\xi_x}Q_{2,y}$$

which by the Schur's complement are equavalent to the following LMI's:

$$\boxed{\begin{bmatrix} Q_{\tilde{z}_2} & P_2 \\ P_2^{\intercal} & I_{n\times n} \end{bmatrix} > 0, \begin{bmatrix} Q_{\xi_x} & Y \\ Y^{\intercal} & I_{n\times n} \end{bmatrix} > 0} \tag{13.11}$$

Both matrices $Q_{\tilde{z}_2}$ and Q_{ξ_x} naturally can be incorporated into the corresponding block-diagonals of matrix $W_{\alpha,\varepsilon}$ (13.7), obtaining

$$W_{\alpha,\varepsilon} \leq \bar{W}_{\alpha,\varepsilon} \tag{13.12}$$

where

$$\boxed{\begin{array}{c} \bar{W}_{\alpha,\varepsilon} = \\ \begin{bmatrix} \begin{matrix} P_1B_0^{\perp}A\left(B_0^{\perp}\right)^{\intercal} + \\ \left[P_1B_0^{\perp}A\left(B_0^{\perp}\right)^{\intercal}\right]^{\intercal} \\ +\alpha P_1 \\ +\varepsilon\left(\tilde{Q}_x+\tilde{Q}_y\right) \end{matrix} & \begin{matrix} P_1B_0^{\perp}A\left(B_0'\right)^{\intercal} + \\ \left[P_2B_0'A\left(B_0^{\perp}\right)^{\intercal}\right]^{\intercal} \\ +\left[YC\left(B_0^{\perp}\right)^{\intercal}\right]^{\intercal} \end{matrix} & P_1B_0^{\perp} & 0_{(2n-\tilde{m})\times k} \\ \begin{matrix} \left[P_1B_0^{\perp}A\left(B_0'\right)^{\intercal}\right]^{\intercal} \\ +P_2B_0'A\left(B_0^{\perp}\right)^{\intercal} \\ +YC\left(B_0^{\perp}\right)^{\intercal} \end{matrix} & \begin{matrix} P_2B_0'A\left(B_0'\right)^{\intercal} + \\ \left[P_2B_0'A\left(B_0'\right)^{\intercal}\right]^{\intercal} \\ +\left[YC\left(B_0'\right)^{\intercal}\right]^{\intercal} + \\ YC\left(B_0'\right)^{\intercal} +\alpha P_2 \\ +Q_{\tilde{z}_2} \end{matrix} & 0_{\tilde{m}\times n} & \begin{matrix} P_2B_0' \\ +P_2Y \end{matrix} \\ \left(P_1B_0^{\perp}\right)^{\intercal} & 0_{n\times\tilde{m}} & \begin{matrix} -\varepsilon I_{n\times n} \\ +Q_{\xi_x} \end{matrix} & 0_{n\times k} \\ 0_{(2n-\tilde{m})\times k} & \left(P_2B_0'\right)^{\intercal} + \left(P_2Y\right)^{\intercal} & 0_{k\times n} & -\varepsilon I_{k\times k} \end{bmatrix}. \end{array}} \tag{13.13}$$

Fulfilling $\bar{W}_{\alpha,\varepsilon} < 0$, in view of (13.12), we guarantee $W_{\alpha,\varepsilon} < 0$.

The optimization procedure for finding the best parameters Θ for the dynamic controller, which provide the minimal size of the attractive ellipsoid, can be formulated now similarly to the problem from lecture 9, namely,

$$\inf_{P_1>0,\ P_2>0,\ Q_{\tilde{z}_2}>0,\ Q_{\xi_x}>0,\ Y,\ \alpha>0,\ \varepsilon>0} \left[\operatorname{tr}(P_1^{-1}) + \operatorname{tr}(P_2^{-1})\right],$$

or equivalently,

$$\boxed{\inf_{P_1>0,\ P_2>0,\ Q_{\tilde{z}_2}>0,\ Q_{\xi_x}>0,\ H_1>0, H_2>0,\ Y,\ \alpha>0,\ \varepsilon>0} \left[\operatorname{tr}(H_1) + \operatorname{tr}(H_2)\right],}$$

subject to LMI's constraints (13.13), (13.11) and additionally (13.14)

$$\boxed{\begin{array}{c} \begin{bmatrix} H_1 & I_{(n-\tilde{m})\times(n-\tilde{m})} \\ I_{(n-\tilde{m})\times(n-\tilde{m})} & P_1 \end{bmatrix} > 0, \\ \begin{bmatrix} H_2 & I_{\tilde{m}\times\tilde{m}} \\ I_{\tilde{m}\times\tilde{m}} & P_2 \end{bmatrix} > 0, \\ \\ H_1 \in \mathbb{R}^{(n-m)\times(n-m)}, H_2 \in \mathbb{R}^{\tilde{m}\times\tilde{m}}, \end{array}} \tag{13.14}$$

obtained from the matrix estimates

$$P_1^{-1} < H_1,\ P_2^{-1} < H_2$$

by the Schur's complement implementation.

13.4 Exercise

Exercise 13.1 *For the system (as in the previous lecture 8)*

$$\left.\begin{array}{c} \dot{x}_1 = -x_1 + x_2 + 0.1 x_1 \operatorname{sign}(x_2), \\ \dot{x}_2 = -x_1 + 0.2 \operatorname{sign}(x_1) + u, \\ y = x_1 + x_2 + \xi_y(t), \\ \\ x_1, x_2 \in \mathbb{R},\ x_0 = (1,1)^{\intercal}, \\ |\xi_y(t)|^2 \le c_{0,y} = 0.01. \end{array}\right\}$$

design the dynamic controller and find its optimal numerical feedback parameters Θ^*. To compare graphically the obtained results with the previous lecture results.