- Movements on a Sling Surface
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- DNN-SM control for the systems of the first order ODE: main result

# Movements on a Sling Surface

In previous Lecture 8 we have considered the DNN system

$$\left.\begin{array}{c} \dfrac{d}{dt}\hat{q}_{1,t} = \hat{q}_{2,t}, \\ \\ \dfrac{d}{dt}\hat{q}_{2,t} = f^{*}_{NN,t}\left(\hat{q}_{1,t}, \hat{q}_{2,t}\right) + u_t \ , \end{array}\right\} \tag{1}$$

and selected the Lyapunov function $V_t$ as

$$\begin{array}{c} V_t = \dfrac{1}{2} s_t^{\intercal} s_t, \\ \\ s_t = \Delta_t \end{array}$$

where $\Delta_t = \hat{q}_{1,t} - q^{*}_{1,t} \in R^n$ is DNNO-tracking error. Our aim was to gurantee $s_t = \Delta_t \underset{t\to\infty}{\to} 0$.

# Sliding variable

Let us consider a new $s_t$ given as

$$\boxed{s_t = \dot{\Delta}_t + \alpha\Delta_t, \alpha > 0.} \tag{2}$$

Our aim again to obtain the property

$$s_t = \dot{\Delta}_t + \alpha\Delta_t \underset{t\to\infty}{\to} 0,$$

which providesDefune the lyapunov function as

$$\dot{\Delta}_t = -\alpha\Delta_t + s_t$$

or equivalently

$$\boxed{\Delta_t = \Delta_0 e^{-\alpha t} + \int\limits_{\tau=0}^{t} e^{-\alpha(t-\tau)} s_\tau d\tau \underset{t\to\infty}{\to} 0}$$

# Lyapunov function and its time behavior

Define the Lyapunov function as

$$V_t = \frac{1}{2} s_t^{\intercal} s_t = \frac{1}{2} \left( \dot{\Delta}_t + \alpha \Delta_t \right)^{\intercal} \left( \dot{\Delta}_t + \alpha \Delta_t \right)$$

which dynamics on the trajectories of the system

$$\left. \begin{array}{c} \dot{\Delta}_{1,t} = \Delta_{2,t}, \\ \\ \dot{\Delta}_{2,t} = f_{NN,t}^{*} - q_{2,t}^{*} + u_t \ , \end{array} \right\} \qquad (3)$$

$$\dot{V}_t = \left( \dot{\Delta}_t + \alpha \Delta_t \right)^{\intercal} \left( \ddot{\Delta}_t + \alpha \dot{\Delta}_t \right) = s_t^{\intercal} \left( f_{NN,t}^{*} - q_{2,t}^{*} + u_t + \alpha \Delta_{2,t} \right)$$

# Lyapunov function and its time behavior

Select $u_t$ fulfilling

$$f^*_{NN,t} - q^*_{2,t} + u_t + \alpha\Delta_{2,t} = -k\mathrm{SIGN}s_t, \;\; k = \mathrm{const} > 0$$

or equivalently, satisfying the following ODE

$$\boxed{u_t = -k\mathrm{SIGN}s_t - f^*_{NN,t} + q^*_{2,t} - \alpha\Delta_{2,t},} \tag{4}$$

which lieads to

$$\dot{V}_t = \left(\dot{\Delta}_t + \alpha\Delta_t\right)^\intercal \left(\ddot{\Delta}_t + \alpha\dot{\Delta}_t\right) =$$
$$s_t^\intercal \left(\dot{f}^*_{NN,t} + \dot{u}_t + \alpha\left[f^*_{NN,t} + u_t\right]\right) = -ks_t^\intercal \mathrm{SIGN}s_t$$

Since

$$s_t^\intercal \mathrm{SIGN}s_t = \sum_{i=1}^{n} s_{i,t}\mathrm{sign}s_{i,t} = \sum_{i=1}^{n} |s_{i,t}| \geq \|s_t\|$$

we get

$$\boxed{\dot{V}_t \leq -k\|s_t\| = -\sqrt{2}k\sqrt{V_t}.}$$

# Lyapunov function and its time behavior

Solution of differential inequality

$$\dot{V}_t \leq -k \left\| s_i \right\| = -\sqrt{2} k \sqrt{V_t}$$

is

$$\frac{dV_t}{\sqrt{V_t}} \leq -\sqrt{2} k dt \Leftrightarrow 2d\left(\sqrt{V_t}\right) \leq -\sqrt{2} k dt$$

$$\sqrt{V_t} - \sqrt{V_0} \leq -\frac{k}{\sqrt{2}} t \Leftrightarrow 0 \leq \sqrt{V_t} \leq \sqrt{V_0} - \frac{k}{\sqrt{2}} t$$

Reaching time $t_{reach}$ when we obtain the sliding surface $s_t = 0$ is

$$\boxed{t_{reach} := \left\{ t : s_t = 0 \Leftrightarrow \sqrt{V_0} - \frac{k}{\sqrt{2}} t = 0 \right\} = \frac{\sqrt{2V_0}}{k} = \frac{\left\| s_0 \right\|}{k}}$$

# Finite reaching time

## Corollary

*The* **Sliding Mode Control** $u_t$ *designed as (4)*

$$\boxed{u_t = -k\mathrm{SIGN}s_t - f^*_{NN,t} + q^*_{2,t} - \alpha\Delta_{2,t}} \tag{5}$$

*with* $\alpha > 0$, $k > 0$ *provides the dynamics*

$$\boxed{s_t = \dot{\Delta}_t + \alpha\Delta_t = 0}$$

*Since on the sliding surface* $\dot{\Delta} + \alpha\Delta = 0$ *for all*

$$\boxed{t \geq t_{reach} = \frac{\|s_0\|}{k} = \frac{\left\|\dot{\Delta}_0 + \alpha\Delta_0\right\|}{k},}$$

*and it (dynamics)* **does not depend** *on neither the applied DNN nor external perturbations.*

# No chattering in control actions

**Fact**

*Since the ODE (5) is, in fact, a low pass filter* **is required to avoid the chattering effect** *in the control action behavior.*

# DNN-SM control for the systems of the first order ODE

Recall (see Lecture 6 and 7) that for these systems DNNN is

$$\frac{d}{dt}\hat{x}_t = f_{NN}\left(\hat{x}_t, t\right) + B_{NN}\left(\hat{x}_t, t\right) u_t,$$
$$f_{NN}\left(\hat{x}_t, t\right) := A\hat{x}_t + L\left[y_t - C\hat{x}_t\right] + W_{0,t}\varphi\left(\hat{x}_t\right),$$
$$B_{NN,t} := B + W_{1,t}\psi\left(\hat{x}_t\right).$$

To design the feedback control, providing $\Delta_t := \hat{x}_t - x_t^* \underset{t\to\infty}{\to} 0$, we can select the Lyapunov function again as

$$V_t = \frac{1}{2}s_t^\intercal \mathcal{M}_t s_t, \;\; s_t = \Delta_t, \;\; \mathcal{M}_t = \mathcal{M}_t^\intercal \geq 0$$
$$\dot{V}_t = s_t^\intercal \mathcal{M}_t \dot{s}_t + \frac{1}{2}s_t^\intercal \dot{\mathcal{M}}_t s_t = s_t^\intercal \mathcal{M}_t \left(\frac{d}{dt}\hat{x}_t - \dot{x}_t^*\right) + \frac{1}{2}s_t^\intercal \dot{\mathcal{M}}_t s_t$$
$$= s_t^\intercal\left(g_t + \mathcal{M}_t B_{NN,t} u_t\right),$$

where

$$g_t := \mathcal{M}_t\left[f_{NN}\left(\hat{x}_t, t\right) - \dot{x}_t^*\right] + \frac{1}{2}\dot{\mathcal{M}}_t s_t$$

# DNN-SM control for the systems of the first order ODE

Let us take

$$\boxed{\mathcal{M}_t := \left( B_{NN,t} B_{NN,t}^{\intercal} \right)^{+}}$$

and

$$\boxed{\begin{array}{c} u_t := -\alpha_t B_{NN,t}^{\intercal} \mathcal{M}_t \mathrm{SIGN}\left( \mathcal{M}_t s_t \right), \\ \\ \alpha_t = \left\| \left[ f_{NN}\left( \hat{x}_t, t \right) - \dot{x}_t^{*} \right] \right\| + \beta_t . \\ \beta_t = \dfrac{s_t^{\intercal} \dot{\mathcal{M}}_t s_t}{2 \left\| \mathcal{M}_t s_t \right\|} + \varrho, \;\; \varrho > 0. \end{array}}$$

Then

$$\begin{gathered}
\dot{V}_t = s_t^{\intercal}\left(g_t + \mathcal{M}_t B_{NN,t} u_t\right) = \\
s_t^{\intercal}\left(g_t\text{-}\alpha_t\left[\mathcal{M}_t B_{NN,t} B_{NN,t}^{\intercal} \mathcal{M}_t\right] \mathrm{SIGN}\left(\mathcal{M}_t s_t\right)\right) = \\
s_t^{\intercal}\mathcal{M}_t\left[f_{NN}\left(\hat{x}_t, t\right) - \dot{x}_t^*\right] + s_t^{\intercal}\frac{1}{2}\dot{\mathcal{M}}_t s_t - \alpha_t\left(\mathcal{M}_t s_t\right)^{\intercal}\mathrm{SIGN}\left(\mathcal{M}_t s_t\right) \\
\leq \left\|\mathcal{M}_t s_t\right\|\left\|\left[f_{NN}\left(\hat{x}_t, t\right) - \dot{x}_t^*\right]\right\| - \alpha_t\sum_{i=1}^{n}\left|\left(\mathcal{M}_t s_t\right)_i\right| + s_t^{\intercal}\frac{1}{2}\dot{\mathcal{M}}_t s_t \\
\leq \left\|\mathcal{M}_t s_t\right\|\left\|\left[f_{NN}\left(\hat{x}_t, t\right) - \dot{x}_t^*\right]\right\| - \alpha_t\left\|\mathcal{M}_t s_t\right\| + s_t^{\intercal}\frac{1}{2}\dot{\mathcal{M}}_t s_t \\
= \left\|\mathcal{M}_t s_t\right\|\left(\left\|\left[f_{NN}\left(\hat{x}_t, t\right) - \dot{x}_t^*\right]\right\| - \alpha_t\right) + s_t^{\intercal}\frac{1}{2}\dot{\mathcal{M}}_t s_t \\
= -\beta_t\left\|\mathcal{M}_t s_t\right\| + s_t^{\intercal}\frac{1}{2}\dot{\mathcal{M}}_t s_t \\
= -\left\|\mathcal{M}_t s_t\right\|\left(\beta_t - s_t^{\intercal}\frac{\dot{\mathcal{M}}_t s_t}{2\left\|\mathcal{M}_t s_t\right\|}\right) = -\varrho\left\|\mathcal{M}_t s_t\right\| < 0.
\end{gathered}$$

Integration gives

$$V_t - V_0 \leq -\varrho \int_{\tau=0}^{t} \|\mathcal{M}_\tau s_\tau\| \, d\tau \Longleftrightarrow \int_{\tau=0}^{t} \|\mathcal{M}_\tau s_\tau\| \, d\tau \leq$$

$$\frac{1}{\varrho}(V_0 - V_t) \leq \frac{V_0}{\varrho} \Longleftrightarrow \int_{\tau=0}^{\infty} \|\mathcal{M}_\tau s_\tau\| \, d\tau \leq \frac{V_0}{\varrho} < \infty,$$

$$\exists t_k : \|\mathcal{M}_{t_k} s_{t_k}\| \underset{k \to \infty}{\to} 0 \Leftrightarrow \mathcal{M}_{t_k} s_{t_k} \underset{k \to \infty}{\to} 0 \Leftrightarrow$$

$$V_{t_k} \underset{k \to \infty}{\to} 0 \text{ if } s_{t_k} \text{ is bounded,}$$

$$V_t \underset{k \to \infty}{\to} V_* \Leftrightarrow V_* = 0.$$

# DNN-SM control for the systems of the first order ODE: main result

## Theorem

*The tracking error* $\Delta_t := \hat{x}_t - x_t^*$ *of the controlled DNN, given by*

$$\boxed{\begin{array}{c} \dfrac{d}{dt}\hat{x}_t = f_{NN}\left(\hat{x}_t, t\right) + B_{NN}\left(\hat{x}_t, t\right) u_t, \\ f_{NN}\left(\hat{x}_t, t\right) := A\hat{x}_t + L\left[y_t - C\hat{x}_t\right] + W_{0,t}\varphi\left(\hat{x}_t\right), \\ B_{NN,t} := B + W_{1,t}\psi\left(\hat{x}_t\right), \\ u_t := -\alpha_t B_{NN,t}^{\intercal} M_t SIGN\left(\mathcal{M}_t s_t\right), \; M_t := \left(B_{NN,t}B_{NN,t}^{\intercal}\right)^{+}, \\ \alpha_t = \left\|\left[f_{NN}\left(\hat{x}_t, t\right) - \dot{x}_t^*\right]\right\| + \frac{s_t^{\intercal}\dot{\mathcal{M}}_t s_t}{2\left\|\mathcal{M}_t s_t\right\|} + \varrho, \; \varrho > 0, \end{array}}$$

*fulfilles the "***property of asymptotic convergence***"*

$$\boxed{V_t = \frac{1}{2}\Delta_t^{\intercal} M_t \Delta_t \underset{k\to\infty}{\longrightarrow} 0.}$$

# DNN-SM control for the systems of the first order ODE: main result

**Remark**

*The values $\dot{\mathcal{M}}_t$ can be calculated using the Super-Twist differentiator.*