Lecture 7: DNN Control for Mechanical Systems

Plan of presentation

- Lagrangian models
- Expression for a mechanical system driven by DC-motor
- Special form of DNN
- Analysis of Attractive Ellipsoid
- Sliding Mode Control for DNN
- Analysis of workability

Lagrangian models

- For the systems of the fiest order

$$\left.\begin{array}{c} \dot{x}_t = f(x_t, t) + g(x_t, t) u_t + \xi_t, \ x_0 \text{ is given,} \\ y_t = C x_t + \eta_t, \end{array}\right\}$$

we considered the following structure of DNN:

$$\frac{d}{dt}\hat{x}_t = A\hat{x}_t + Bu_t + L[y_t - C\hat{x}_t] + W_{0,t}\varphi(\hat{x}_t) + W_{1,t}\psi(\hat{x}_t) u_t$$

which can be represented as

$$\frac{d}{dt}\hat{x}_t = f_{NN}(\hat{x}_t, t) + B_{NN}(\hat{x}_t, t) u_t,$$

with some initial conditions $\hat{x}_0$, where

$$\begin{array}{c} f_{NN}(\hat{x}_t, t) := A\hat{x}_t + L[y_t - C\hat{x}_t] + W_{0,t}\varphi(\hat{x}_t), \\ B_{NN}(\hat{x}_t, t) := B + W_{1,t}\psi(\hat{x}_t). \end{array}$$

Lagrangian models

- Lagrangian mechanical models have the form

$$\boxed{\frac{d}{dt}\frac{\partial}{\partial \dot{q}}L - \frac{\partial}{\partial q}L = Q_{non-pot}, \quad L = T - V}$$

which in the open format is (omitting time-dependence)

$$\boxed{\begin{array}{c} D(q)\,\ddot{q} + C(q,\dot{q})\,\dot{q} + g(q) = \tau + \vartheta, \\ D(q) = D^{\intercal}(q) > 0 \forall q \in R^{n}, \end{array}}$$

or as

$$\boxed{\left.\begin{array}{c} \dot{q}_1 = q_2 \\ \dot{q}_2 = f_{unc}(q,\dot{q}) + D^{-1}(q)\,\tau \\ \\ f_{unc}(q,\dot{q}) := \text{-}D^{-1}(q)\,C(q,\dot{q})\,\dot{q}\text{-}D^{-1}(q)\,g(q) + D^{-1}(q)\,\vartheta \end{array}\right\}} \tag{1}$$

$\vartheta \in R^{n}$ is the disturbance (or noise) vector.

Available information

- Output $y_t = Cq_t + \eta_t$ is **measurable**, the matrix $D(q)$ is **known**.
- Matrix $C(q, \dot{q})$ (frictions and etc.) and vector $g(q)$ (potential forces) admit to be **unknown**.
- Torque force τ is activated by n-independent **Permanent Magnet DC (PMDC) motors**:

$$\boxed{\tau_t = WK_a I_{at}, \ \ L_a \dot{I}_{at} + R_a I_{at} + K_e W^{\intercal} \dot{q}_t = v_{at} \,,} \tag{2}$$

$I_{at} \in R^n$ - the armature current vector, $L_a = \mathrm{diag}\{L_{a1}, L_{a2}, ..., L_{an}\}$ and $R_a = \mathrm{diag}\{R_{a1}, R_{a2}, ..., R_{an}\}$ are the armature inductances and resistants positive matrices, respectively, $W \in R^{n \times n}$ is the electromotive force constant invertable matrix (possibly taking into account the gear ratios of the motors), $K_a \in R^{n \times n}$ - the direct - electromotive forces constants matrix, $K_e = \mathrm{diag}\{K_{e1}, K_{e2}, ..., K_{en}\} \in R^{n \times n}$- the back-electromotive forces constants with positive elements, $\vartheta_t \in R^n$ is the disturbance (or uncertainty) vector, $v_{at} \in R^n$ - the armature **voltage vector**, which below is considered as **a control** to be designed to obtain a desired behavior.

Suppose that q_t, $\dot{q}_t$ and I_{at} are available (or estimated) on-line. From (2) it follows

$$I_{at}\text{-}I_{at_0} = -L_a^{-1} \int_{\tau=t_0}^{t} \left[R_a I_{a\tau} + K_e W^\intercal \dot{q}_\tau - v_{a\tau}\right] d\tau \tag{3}$$

($t_0 \geq 0$ is any fixed time), and selecting (neglecting the Joule effect, related to the dependence of the winding motor resistance)

$$v_{at} = R_a I_{a\tau} + K_e W^\intercal \dot{q}_t + L_a K_a^{-1} W^{-1} \tilde{v}_{at},$$

the relation (3) becomes

$$I_{at} = I_{at_0} - K_a^{-1} W^{-1} \int_{\tau=t_0}^{t} \tilde{v}_{a\tau} d\tau. \tag{4}$$

Final expression for a mechanical system driven by DC-motor

Substituting (4) into (1) gives

$$\boxed{\left.\begin{aligned}\dot{q}_1 &= q_2, \\ \dot{q}_2 &= f_{unc}\left(q_1, q_2\right) + u_t\,.\end{aligned}\right\}} \tag{5}$$

$$\boxed{u_t := -\int\limits_{\tau=t_0}^{t} \tilde{v}_{a\tau}\, d\tau} \tag{6}$$

with $I_{at_0} = 0$.

Structure of DNN model

According to the physical model representation (5) define DNN model as

$$\left.\begin{array}{c}\frac{d}{dt}\hat{q}_{1,t}=\hat{q}_{2,t}\\[2ex]\frac{d}{dt}\hat{q}_{2,t}=A\hat{q}_{2,t}+L\left[y_t-C\begin{pmatrix}\hat{q}_{1,t}\\ \hat{q}_{2,t}\end{pmatrix}\right]+W_{0,t}\varphi\left(\hat{q}_{1,t}\right)+u_t\,.\end{array}\right\}\tag{7}$$

Storage function

For the processes

$$\delta_{1,t} = \hat{q}_{1,t} - q_{1,t}, \delta_{2,t} = \dot{\delta}_{1,t} = \frac{d}{dt}\hat{q}_{1,t} - \dot{q}_{1,t},$$

satisfying

$$\begin{gathered} \dot{\delta}_{1,t} = \delta_{2,t} \\ \dot{\delta}_{2,t} = A\hat{q}_{2,t} + L\left[y_t - C\begin{pmatrix} \hat{q}_{1,t} \\ \hat{q}_{2,t} \end{pmatrix}\right] + W_{0,t}\varphi\left(\hat{q}_{1,t}\right) \text{-} f_{unc}\left(q_1, q_2\right) \end{gathered}$$

Define the storage function as

$$\left.\begin{gathered} V\left(\delta_{1,t}, \delta_{2,t}\right) = \delta_{1,t}^{\intercal} P_1 \delta_{1,t} + \delta_{2,t}^{\intercal} P_2 \delta_{2,t} + \\ \frac{1}{2}\mathrm{tr}\left\{\left(W_{0,t} - W_0^*\right)^{\intercal} \Lambda \left(W_{0,t} - W_0^*\right)\right\} \\ \\ P_1 = P_1^{\intercal} > 0,\ P_2 = P_2^{\intercal} > 0,\ \Lambda = \Lambda^{\intercal} > 0 \end{gathered}\right\} \tag{8}$$

From (8) we get

$$\dot{V}\left(\delta_{1,t},\delta_{2,t}\right)=2\delta_{1,t}^{\intercal}P_{1}\delta_{2,t}+2\delta_{2,t}^{\intercal}P_{2}\dot{\delta}_{2,t}+\operatorname{tr}\left\{\left(W_{0,t}-W_{0}^{*}\right)^{\intercal}\Lambda\dot{W}_{0,t}\right\}$$
$$=2\delta_{1,t}^{\intercal}P_{1}\delta_{2,t}+2\delta_{2,t}^{\intercal}P_{2}\left(A\left[\hat{q}_{2,t}-q_{2,t}\right]-LC\begin{pmatrix}\delta_{1,t}\\ \delta_{2,t}\end{pmatrix}+W_{0}^{*}\varphi\left(\hat{q}_{1,t}\right)\right.$$
$$\left.\left(W_{0,t}-W_{0}^{*}\right)\varphi\left(\hat{q}_{1,t}\right)+\left[Aq_{2,t}-f_{unc}\left(q,\dot{q}\right)\right]\right)+\operatorname{tr}\left\{\left(W_{0,t}-W_{0}^{*}\right)^{\intercal}\Lambda\dot{W}_{0,t}\right\}$$
$$=\begin{pmatrix}\delta_{1,t}\\ \delta_{2,t}\\ \varphi\left(\hat{q}_{1,t}\right)\\ Aq_{2,t}-f_{unc}\left(q_{1},q_{2}\right)\end{pmatrix}^{\intercal}S_{0}\begin{pmatrix}\delta_{1,t}\\ \delta_{2,t}\\ \varphi\left(\hat{q}_{1,t}\right)\\ Aq_{2,t}-f_{unc}\left(q_{1},q_{2}\right)\end{pmatrix}+$$
$$2\delta_{2,t}^{\intercal}P_{2}\left(W_{0,t}-W_{0}^{*}\right)\varphi\left(\hat{q}_{1,t}\right)+\operatorname{tr}\left\{\left(W_{0,t}-W_{0}^{*}\right)^{\intercal}\Lambda\dot{W}_{0,t}\right\}$$

where

$$S_{0}:=\begin{bmatrix}0_{n\times n} & P_{1}-P_{2}LC & 0_{n\times n} & 0_{n\times n}\\ P_{1}-C^{\intercal}L^{\intercal}P_{2} & \begin{matrix}P_{2}A+A^{\intercal}P_{2}\\ -P_{2}LC-C^{\intercal}L^{\intercal}P_{2}\end{matrix} & P_{2}W_{0}^{*} & P_{2}\\ 0_{n\times n} & \left(W_{0}^{*}\right)^{\intercal}P_{2} & 0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & P_{2} & 0_{n\times n} & 0_{n\times n}\end{bmatrix}$$

Right-hand side reorganization

Defining the extended vector

$$z_t := \left(\delta_{1,t}^{\intercal}, \quad \delta_{2,t}^{\intercal}, \quad [\varphi(\hat{q}_{1,t})]^{\intercal}, \quad (Aq_{2,t} - f_{unc}(q_1, q_2))^{\intercal} \right)^{\intercal}$$

we get with $\alpha > 0$

$$\dot{V}_t = z_t^{\intercal} \begin{bmatrix} (\alpha - \varepsilon) P_1 & P_1 - P_2 LC & 0_{n\times n} & 0_{n\times n} \\ P_1 - C^{\intercal} L^{\intercal} P_2 & \begin{array}{c} \alpha P_2 + P_2 A + A^{\intercal} P_2 \\ -P_2 LC - C^{\intercal} L^{\intercal} P_2 \end{array} & P_2 W_0^* & P_2 \\ 0_{n\times n} & (W_0^*)^{\intercal} P_2 & -\varepsilon I_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & P_2 & 0_{n\times n} & -\varepsilon I_{n\times n} \end{bmatrix} z_t$$
$$+\mathrm{tr}\left\{ (W_{0,t} - W_0^*)^{\intercal} \Lambda \left[\tfrac{\alpha}{2} (W_{0,t} - W_0^*) + \dot{W}_{0,t} + 2\Lambda^{-1} \varphi(\hat{q}_{1,t}) \delta_{2,t}^{\intercal} P_2 \right] \right\} +$$
$$\varepsilon \left\| (W_{0,t} - W_0^*) \varphi(\hat{q}_{1,t}) \right\|^2 + \varepsilon \left\| (Aq_{2,t} - f_{unc}(q_1, q_2)) \right\|^2 - \alpha V_t + \varepsilon \delta_{1,t}^{\intercal} P_1 \delta_{1,t}$$

The upper estimate for the last term

$$\left\|\left(Aq_{2,t}-f_{unc}\left(q_1,q_2\right)\right)\right\|^2 \leq c_0+c_1\left\|q_1\right\|^2+c_2\left\|q_2\right\|^2 =$$
$$c_0+c_1\left\|\left(q_{1,t}-\hat{q}_{1,t}\right)+\hat{q}_{1,t}\right\|^2+c_2\left\|\left(q_{2,t}-\hat{q}_{2,t}\right)+\hat{q}_{2,t}\right\|^2 \leq$$
$$c_0+2c_1\left(\left\|\delta_{1,t}\right\|^2+\left\|\hat{q}_{1,t}\right\|^2\right)+2c_2\left(\left\|\delta_{2,t}\right\|^2+\left\|\hat{q}_{2,t}\right\|^2\right)$$

which leads to

$$\dot{V}_t \leq z_t^{\intercal} S z_t - \alpha V_t + \varepsilon c_0 + \varepsilon \rho_t + \operatorname{tr}\left\{\left(W_{0,t}-W_0^*\right)^{\intercal}\Lambda\times\right.$$
$$\left.\left(\frac{\alpha}{2}\left(W_{0,t}-W_0^*\right)+\dot{W}_{0,t}+\varepsilon\Lambda^{-1}\left(W_{0,t}-W_0^*\right)\varphi\left(\hat{q}_{1,t}\right)\varphi^{\intercal}\left(\hat{q}_{1,t}\right)\right)\right\}$$

$$S := \begin{bmatrix} \begin{matrix}(\alpha-\varepsilon)P_1 \\ +2\varepsilon c_1 I_{n\times n}\end{matrix} & P_1-P_2LC & 0_{n\times n} & 0_{n\times n} \\ P_1-C^{\intercal}L^{\intercal}P_2 & \begin{matrix}\alpha P_2+P_2A+A^{\intercal}P_2-P_2LC \\ -C^{\intercal}L^{\intercal}P_2+2\varepsilon c_2 I_{n\times n}\end{matrix} & P_2W_0^* & P_2 \\ 0_{n\times n} & \left(W_0^*\right)^{\intercal}P_2 & -\varepsilon I_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & P_2 & 0_{n\times n} & -\varepsilon I_{n\times n} \end{bmatrix}$$

$$\rho_t := \left[\delta_{1,t}^{\intercal}P_1\delta_{1,t}+c_0+2c_1\left\|\hat{q}_{1,t}\right\|^2+2c_2\left\|\hat{q}_{2,t}\right\|^2\right]$$

Learning law

The term ρ_t can be expressed as

$$\varepsilon\rho_t = \varepsilon\rho_t \frac{\mathrm{tr}\left\{\left(W_{0,t}-W_0^*\right)^\intercal \Lambda \left(W_{0,t}-W_0^*\right)\right\}}{\mathrm{tr}\left\{\left(W_{0,t}-W_0^*\right)^\intercal \Lambda \left(W_{0,t}-W_0^*\right)\right\}} =$$
$$\mathrm{tr}\left\{\left(W_{0,t}-W_0^*\right)^\intercal \left[\frac{\varepsilon\rho_t \Lambda \left(W_{0,t}-W_0^*\right)}{\mathrm{tr}\left\{\left(W_{0,t}-W_0^*\right)^\intercal \Lambda \left(W_{0,t}-W_0^*\right)\right\}}\right]\right\}$$

and can be added to the last term:

$$\dot{V}_t \leq z_t^\intercal S z_t - \alpha V_t + \varepsilon c_0 + \mathrm{tr}\left\{\left(W_{0,t}-W_0^*\right)^\intercal \Lambda \times\right.$$

$$\left[\tfrac{\alpha}{2}\left(W_{0,t}-W_0^*\right) + \dot{W}_{0,t} + \varepsilon\Lambda^{-1}\left(W_{0,t}-W_0^*\right)\varphi\left(\hat{q}_{1,t}\right)\varphi^\intercal\left(\hat{q}_{1,t}\right) + \right.$$

$$\left.\left. + \frac{\varepsilon\rho_t\left(W_{0,t}-W_0^*\right)}{\mathrm{tr}\left\{\left(W_{0,t}-W_0^*\right)^\intercal \Lambda \left(W_{0,t}-W_0^*\right)\right\}}\right]\right\}$$

Learning law

It the following Learning law holds

$$\boxed{\begin{array}{c}\dot{W}_{0,t}=-\frac{\alpha}{2}\left(W_{0,t}-W_0^*\right)-\varepsilon\Lambda^{-1}\left(W_{0,t}-W_0^*\right)\varphi\left(\hat{q}_{1,t}\right)\varphi^{\intercal}\left(\hat{q}_{1,t}\right)\\ \\ -\dfrac{\varepsilon\rho_t\left(W_{0,t}-W_0^*\right)}{\operatorname{tr}\left\{\left(W_{0,t}-W_0^*\right)^{\intercal}\Lambda\left(W_{0,t}-W_0^*\right)\right\}},\end{array}} \quad (9)$$

and

$$S\leq 0,$$

then

$$\boxed{\dot{V}_t\leq-\alpha V_t+\varepsilon c_0}$$

Attractive ellipsoid

Theorem (on Attractive Ellipsoid)

If the Learning process (9) takes place and $S \leq 0$, then

$$\boxed{\limsup_{t\to\infty} \leq \limsup_{t\to\infty} \frac{\alpha}{\varepsilon c_0} V_t \leq 1}$$

or equivalently, $\frac{\alpha}{\varepsilon c_0} \left(\delta_{1,t}^{\intercal} P_1 \delta_{1,t} + \delta_{2,t}^{\intercal} P_2 \delta_{2,t}\right)$

$$\left.\begin{array}{c} \limsup_{t\to\infty} \begin{pmatrix} \delta_{1,t} \\ \delta_{2,t} \end{pmatrix}^{\intercal} \underbrace{\left(\frac{\alpha}{\varepsilon c_0} \begin{bmatrix} P_1 & 0_{n\times n} \\ 0_{n\times n} & P_2 \end{bmatrix} \right)}_{P_{attr}} \begin{pmatrix} \delta_{1,t} \\ \delta_{2,t} \end{pmatrix} = \\ \limsup_{t\to\infty} \begin{pmatrix} \delta_{1,t} \\ \delta_{2,t} \end{pmatrix}^{\intercal} P_{attr} \begin{pmatrix} \delta_{1,t} \\ \delta_{2,t} \end{pmatrix} \leq 1 \end{array}\right\} \quad (10)$$